Complexity of parametric integration in various smoothness classes

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Abstract

We continue the complexity analysis of parametric definite and indefinite integration given by the authors in [2]. Here we consider anisotropic classes of functions, including certain classes with dominating mixed derivatives. Our analysis is based on a multilevel Monte Carlo method developed in [2] and we obtain the order of the deterministic and randomized n-th minimal errors (in some limit cases up to logarithms). Furthermore, we compare the rates in the deterministic and randomized setting to assess the gain reached by randomization.

1 Introduction

The complexity of definite parametric integration was studied in [10], [6], and [16], while in [2] the complexity of both definite and indefinite parametric integration was considered. Parametric definite integration is a problem intermediate between integration and approximation. Parametric indefinite integration can be viewed as a model for the solution of parametric initial value problems in the sense that it is a partial, but typical case, and some of the methods developed here will be used in the study of parametric initial value problems, see [3].

This paper is a continuation of [2] and we study both definite and indefinite integration. So far definite parametric integration was considered only for isotropic classes and, in [6], for a specific anisotropic class (Sobolev case with no smoothness in the integration variable). Indefinite parametric integration was only studied for C^r. In [2] we gave a general (multilevel) scheme for Banach space valued integration of functions belonging to

$$C^r(X) \cap C^{r_1}(Y), \tag{1}$$

where X and Y are Banach spaces such that Y is continuously embedded into X, from which the upper bounds for parametric integration in the C^r-case were derived.

In the present paper we further explore the range given in (1) by considering classes of functions with dominating mixed derivatives and other types of non-isotropic smoothness. In contrast to the C^r case, these classes allow to treat different smoothnesses for the parameter dependence and for the basic (nonparametric) integration problem. We want to understand the typical behaviour of the complexity in these classes and the relation between the deterministic and randomized setting, this way clarifying in which cases and to which extend randomized methods are superior to deterministic ones.

The paper is organized as follows. In Section 3 we recall the needed algorithms and results for Banach space valued definite and indefinite integration from [2]. In Section 4 we consider parametric definite and indefinite integration and obtain the main results. Applications to various smoothness classes are given in Section 5, together with some comments on the relation between the deterministic and the randomized setting.

2 Preliminaries

We denote $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Given Banach spaces X, Y, we let $\mathscr{L}(X, Y)$ be the space of bounded linear operators from X to Y, equipped with the usual norm, and we write $\mathscr{L}(X)$ if $X = Y$. The dual space of X is denoted by X^*, the identity mapping on X by I_X, and the closed unit ball by B_X. The norm of X is denoted by $\|\cdot\|$, other norms are distinguished by subscripts. We assume all considered Banach spaces to be defined over the same scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

We often use the same symbol for possibly different constants. Given two sequences of nonnegative reals $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, the notation $a_n \preceq b_n$ means that there are constants $c > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $a_n \leq c b_n$. Moreover, we write $a_n \asymp b_n$ if $a_n \preceq b_n$ and $b_n \preceq a_n$. We also use the notation $a_n \asymp_{\log} b_n$ if there are constants $c_1, c_2 > 0$, $n_0 \in \mathbb{N}$, and $\theta_1, \theta_2 \in \mathbb{R}$ with $\theta_1 \leq \theta_2$ such that for all $n \geq n_0$

$$c_1 b_n (\log(n+1))^{\theta_1} \leq a_n \leq c_2 b_n (\log(n+1))^{\theta_2}.$$

Throughout the paper log means $\log_2$.

For $1 \leq p \leq 2$ a Banach space X is called to be of (Rademacher) type p, if there is a constant $c \geq 0$ such that for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$

$$\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^p \leq c^p \sum_{k=1}^n \|x_i\|^p, \tag{2}$$

with $(\varepsilon_i)_{i=1}^n$ being independent random variables satisfying $\mathbb{P}\{\varepsilon_i = -1\} = \mathbb{P}\{\varepsilon_i = +1\} = 1/2$. The type p constant $\tau_p(X)$ of X is the smallest constant $c \geq 0$ satisfying (2), and $\tau_p(X) = \infty$, if there is no such c. We refer to [11] for background on this notion. The space $L_{p_1}(\mathcal{M}, \mu)$, where $(\mathcal{M}, \mu)$ is an arbitrary measure space and $p_1 < \infty$, is of type p with $p = \min(p_1, 2)$. Furthermore, there is a constant $c > 0$ such that $\tau_2(\ell_\infty^n) \leq c(\log(n+1))^{1/2}$ for all $n \in \mathbb{N}$.

Let $Q = [0,1]^d$ and let $C^r(Q, X)$ denote the space of all r-times continuously differentiable functions $f : Q \to X$ equipped with the norm

$$\|f\|_{C^r(Q,X)} = \max_{\alpha \in \mathbb{N}_0^d,\, |\alpha| \leq r,\, t \in Q} \left\| \frac{\partial^{|\alpha|} f(t)}{\partial t^\alpha} \right\|.$$

For $r = 0$ we write $C^0(Q, X) = C(Q, X)$, which is the space of continuous X-valued functions on Q, and if $X = \mathbb{K}$, we write $C^r(Q)$ and $C(Q)$.

Let $X \otimes Y$ be the algebraic tensor product of Banach spaces X and Y and let $X \otimes_\lambda Y$ be the injective tensor product, defined as the completion of $X \otimes Y$ with respect to the norm

$$\lambda\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sup_{u \in B_{X^*},\, v \in B_{Y^*}} \left| \sum_{i=1}^n \langle x_i, u\rangle \langle y_i, v\rangle \right|.$$

Background on tensor products can be found in [4] and [12]. For Banach spaces X_1, Y_1 and operators $T \in \mathscr{L}(X, X_1)$, $U \in \mathscr{L}(Y, Y_1)$, the algebraic tensor product $T \otimes U : X \otimes Y \to X_1 \otimes Y_1$ extends to a bounded linear operator $T \otimes U \in \mathscr{L}(X \otimes_\lambda Y, X_1 \otimes_\lambda Y_1)$ with

$$\|T \otimes U\|_{\mathscr{L}(X \otimes_\lambda Y, X_1 \otimes_\lambda Y_1)} = \|T\|_{\mathscr{L}(X, X_1)} \|U\|_{\mathscr{L}(Y, Y_1)}.$$

We also recall that for each Banach space X the canonical isometric identification

$$C(Q, X) = X \otimes_\lambda C(Q), \tag{3}$$

holds. It follows that, in particular, for $d > 1$

$$C([0,1]^d) = C([0,1]) \otimes_\lambda \cdots \otimes_\lambda C([0,1]).$$

Based on this, we define for $r, m \in \mathbb{N}$

$$P_m^{r,d} = P_m^{r,1} \otimes \cdots \otimes P_m^{r,1} \in \mathscr{L}(C([0,1]^d)),$$

where $P_m^{r,1} \in \mathscr{L}(C([0,1]))$ denotes composite with respect to the partition of $[0,1]$ into m intervals of length m^{-1} Lagrange interpolation of degree r. Setting $\Gamma_k^d = \left\{\frac{i}{k} : 0 \leq i \leq k\right\}^d$ for $k \in \mathbb{N}$, it follows that $P_m^{r,d}$ interpolates on Γ_{rm}^d. We will use the well-known fact that there are constants $c_1, c_2 > 0$ such that for all $m \in \mathbb{N}$

$$\|P_m^{r,d}\|_{\mathscr{L}(C(Q))} \leq c_1, \qquad \sup_{f \in B_{C^r(Q)}} \|f - P_m^{r,d} f\|_{C(Q)} \leq c_2 m^{-r}. \tag{4}$$

Next we recall some notation from information-based complexity theory [15, 14], see also [7, 8] for the precise notions used here. Let F be a nonempty set, G a normed linear space, $S : F \to G$ an arbitrary mapping, let K be a nonempty set, and let Λ be a set of mappings from F to K. We interpret F as the set of inputs, S as the solution operator, that is, the mapping that sends the input $f \in F$ to the exact solution Sf, and Λ is understood as the class of admissible information functionals. Thus, the tuple $\mathscr{P} = (F, G, S, K, \Lambda)$ describes the abstract numerical problem under consideration. In this paper we always have $K = \mathbb{K}$.

A deterministic algorithm A for $\mathscr{P}$ is a mapping $A : F \to G$, which is built from values of information functionals on $f \in F$ in an adaptive way (all details can be found in [7, 8]). The result of the algorithm Af is the approximation to Sf. The error of A is given by

$$e(S, A, F) = \sup_{f \in F} \|Sf - Af\|_G.$$

Let $\operatorname{card}(A, f)$ be the number of information functionals used by A at input f and put

$$\operatorname{card}(A, F) = \sup_{f \in F} \operatorname{card}(A, f).$$

Now the deterministic n-th minimal error is defined for $n \in \mathbb{N}_0$ by

$$e_n^{\det}(S, F) = \inf_{\operatorname{card}(A,F) \le n} e(S, A, F).$$

A randomized algorithm for $\mathscr{P}$ is a family $A = (A_\omega)_{\omega \in \Omega}$, where $(\Omega, \Sigma, \mathbb{P})$ is a probability space and for each $\omega \in \Omega$, A_ω is a deterministic algorithm. The parameter ω represents the randomness in the algorithm $A = (A_\omega)_{\omega \in \Omega}$. The error of A is given by

$$e(S, A, F) = \sup_{f \in F} \mathbb{E}\, \|Sf - A_\omega f\|_G.$$

Setting

$$\operatorname{card}(A, F) = \sup_{f \in F} \mathbb{E} \operatorname{card}(A_\omega, f),$$

the randomized n-th minimal error is defined for $n \in \mathbb{N}_0$ by

$$e_n^{\mathrm{ran}}(S, F) = \inf_{\operatorname{card}(A,F) \le n} e(S, A, F).$$

So $e_n^{\det}(S, F)$, respectively $e_n^{\mathrm{ran}}(S, F)$, is the minimal possible error among all deterministic, respectively randomized algorithms that use at most n information functionals. Since any deterministic algorithm can be viewed as a special case of a randomized algorithm with a one-point probability space $\Omega = \{\omega_0\}$, we always have $e_n^{\mathrm{ran}}(S, F) \le e_n^{\det}(S, F)$.

3 Banach space valued integration and a multilevel method

Given $r \in \mathbb{N}_0$ and a Banach space X, we introduce the definite integration operator $S_0^X : C(Q,X) \to X$ by

$$S_0^X f = \int_Q f(t)dt \quad (f \in C(Q,X)) \tag{5}$$

and the indefinite integration operator $S_1^X : C(Q,X) \to C(Q,X)$ by

$$(S_1^X f)(t) = \int_{[0,t]} f(u)du \quad (t \in Q, f \in C(Q,X)), \tag{6}$$

where $[0,t] = \prod_{i=1}^d [0,t_i]$, $t = (t_i)_{i=1}^d \in Q$. Let S_ι ($\iota = 0,1$) be the scalar version of S_ι^X, that is, $X = \mathbb{K}$. Then, in the sense of (3), we have $S_\iota^X = I_X \otimes S_\iota$.

First we recall algorithms for the scalar cases of the integration problems (5) and (6). For $r = 0$ and $n \in \mathbb{N}$ the standard Monte Carlo method for definite integration is given by

$$A_{n,\omega}^{0,0} f = \frac{|Q|}{n} \sum_{i=1}^n f(\xi_i(\omega)) \quad (f \in C(Q)),$$

where $\xi_i : \Omega \to Q$ ($i = 1, \dots, n$) are independent, uniformly distributed on Q random variables on some complete probability space $(\Omega, \Sigma, \mathbb{P})$. If $r \ge 1$, we put $k = \lceil n^{1/d} \rceil$ and

$$A_{n,\omega}^{0,r} f = S_0(P_k^{r,d} f) + A_{n,\omega}^{0,0}(f - P_k^{r,d} f),$$

which is the Monte Carlo method with separation of the main part. Finally we set $A_n^{0,r} = \left(A_{n,\omega}^{0,r}\right)_{\omega \in \Omega}$.

For indefinite integration we recall the algorithm from Section 4 of [9]. Let $n \in \mathbb{N}$, and put $m = \left\lceil (n+1)^{\frac{1}{2d-1}} \right\rceil$. For $\bar{l} = (l_1, \dots, l_d) \in \mathbb{N}_0^d$ we define $U_{\bar{l}}$ by

$$U_{\bar{l}} \;=\; (P_{m^{l_1}}^{1,1} - P_{m^{l_1-1}}^{1,1}) \otimes \dots \otimes (P_{m^{l_{d-1}}}^{1,1} - P_{m^{l_{d-1}-1}}^{1,1}) \otimes P_{m^{l_d}}^{1,1},$$

($P_{m^{-1}}^{1,1} := 0$) and set

$$V \;=\; \sum_{\bar{l} \in \mathbb{N}_0^d,\, |\bar{l}| = 2d-1} U_{\bar{l}},$$

where $|\bar{l}| = l_1 + \dots + l_d$. Moreover, we define

$$m^{\bar{l}} = (m^{l_1}, \dots, m^{l_d}), \quad \Gamma_{m^{\bar{l}}} = \Gamma_{m^{l_1}}^1 \times \dots \times \Gamma_{m^{l_d}}^1,$$

and for $\bar{i} = (i_1, \dots, i_d) \in \mathbb{N}^d$ with $1 \le i_k \le m^{l_k}$ $(1 \le k \le d)$

$$Q_{\bar{l},\bar{i}} = \left[\frac{i_1 - 1}{m^{l_1}}, \frac{i_1}{m^{l_1}}\right] \times \cdots \times \left[\frac{i_d - 1}{m^{l_d}}, \frac{i_d}{m^{l_d}}\right].$$

Let $\xi_{\bar{l},\bar{i}} : \Omega \to Q_{\bar{l},\bar{i}}$ $(|\bar{l}| = 2d-1,\, \bar{1} \le \bar{i} \le m^{\bar{l}})$ be independent uniformly distributed on $Q_{\bar{l},\bar{i}}$ random variables on a complete probability space $(\Omega, \Sigma, \mathbb{P})$. Define $g_{\bar{l},\omega} \in \ell_\infty(\Gamma_{m^{\bar{l}}})$ by

$$g_{\bar{l},\omega}(t) = \sum_{\bar{j}: Q_{\bar{l},\bar{j}} \subseteq [0,t]} |Q_{\bar{l},\bar{j}}| f(\xi_{\bar{l},\bar{j}}(\omega)) \quad (t \in \Gamma_{m^{\bar{l}}}),$$

where the sum is set to zero if there is no $\bar{j}$ with $Q_{\bar{l},\bar{j}} \subseteq [0,t]$. Finally we put $A_n^{1,r} = \left(A_{n,\omega}^{1,r}\right)_{\omega \in \Omega}$ with

$$A_{n,\omega}^{1,0} f := \sum_{\bar{l} \in \mathbb{N}_0^d,\, |\bar{l}| = 2d-1} U_{\bar{l}} g_{\bar{l},\omega}$$

and, for $r \ge 1$, with $k = \lceil n^{1/d} \rceil$,

$$A_{n,\omega}^{1,r} f = S_1(P_k^{r,d} f) + A_{n,\omega}^{1,0}(f - P_k^{r,d} f).$$

Now we let $r, r_1 \in \mathbb{N}_0$ and consider integration of functions from the set

$$B_{C^r(Q,X)} \cap B_{C^{r_1}(Q,Y)},$$

where Y is a Banach space continuously embedded into X. We identify elements of Y with their images in X. The following scheme was developed in [2], based on the multilevel Monte Carlo approach from [5, 10]. Let $(T_l)_{l=0}^\infty \subset \mathscr{L}(X)$, $l_0, l_1 \in \mathbb{N}_0$, $l_0 \le l_1$, $n_{l_0}, \dots, n_{l_1} \in \mathbb{N}$, and define an algorithm $A^{(\iota)} = (A_\omega^{(\iota)})_{\omega \in \Omega}$ for $\iota \in \{0, 1\}$ as follows:

$$A_\omega^{(\iota)} = T_{l_0} \otimes A_{n_{l_0},\omega}^{\iota,r} + \sum_{l=l_0+1}^{l_1} (T_l - T_{l-1}) \otimes A_{n_l,\omega}^{\iota,r_1} \quad (f \in C(Q,X)). \tag{7}$$

To state the next result, we need some more notation. Let $J : Y \to X$ be the embedding map, put $G_0(X) = X$, $G_1(X) = C(Q,X)$, and

$$X_l = \mathrm{cl}_X(T_l(X)) \quad (l \in \mathbb{N}_0), \quad X_{l-1,l} = \mathrm{cl}_X((T_l - T_{l-1})(X)) \quad (l \in \mathbb{N}), \tag{8}$$

where cl_X denotes the closure in X. The following is a slight extension of Proposition 3 of [2]. We omit the proof, since it is essentially the same as in [2], except that in (10) for a part of the series the deterministic estimate is applied. This is needed to obtain precise rates in Section 4.

Proposition 3.1. *Let $1 \le p \le 2$, $r, r_1 \in \mathbb{N}_0$, and $\iota \in \{0,1\}$. Then there are constants $c_1, c_2 > 0$ such that for all Banach spaces X, Y, and operators $(T_l)_{l=0}^\infty$ as above, for all $l_0, l_1 \in \mathbb{N}_0$ with $l_0 \le l_1$, and for all $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ the so-defined algorithm $A_\omega^{(\iota)}$ satisfies*

$$
\begin{aligned}
&\sup_{f \in B_{C^r(Q,X)} \cap B_{C^{r_1}(Q,Y)}} \|S_\iota^X f - A_\omega^{(\iota)} f\|_{G_\iota(X)} \\
&\le \|J - T_{l_1} J\|_{\mathscr{L}(Y,X)} + c_1 \|T_{l_0}\|_{\mathscr{L}(X)}\, n_{l_0}^{-r/d} \\
&\quad + c_1 \sum_{l=l_0+1}^{l_1} \|(T_l - T_{l-1})J\|_{\mathscr{L}(Y,X)}\, n_l^{-r_1/d} \quad (\omega \in \Omega) \qquad (9)
\end{aligned}
$$

and for all $l^* \in \mathbb{N}_0$ with $l_0 \le l^* \le l_1$

$$
\begin{aligned}
&\sup_{f \in B_{C^r(Q,X)} \cap B_{C^{r_1}(Q,Y)}} \left(\mathbb{E}\, \|S_\iota^X f - A_\omega^{(\iota)} f\|_{G_\iota(X)}^p\right)^{1/p} \\
&\le \|J - T_{l_1} J\|_{\mathscr{L}(Y,X)} + c_2 \tau_p(X_{l_0}) \|T_{l_0}\|_{\mathscr{L}(X)} n_{l_0}^{-r/d-1+1/p} \\
&\quad + c_2 \sum_{l=l_0+1}^{l^*} \tau_p(X_{l-1,l}) \|(T_l - T_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-r_1/d-1+1/p} \\
&\quad + c_2 \sum_{l=l^*+1}^{l_1} \|(T_l - T_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-r_1/d}. \qquad (10)
\end{aligned}
$$

4 Parametric integration

Let $d_0 \in \mathbb{N}$, $Q_0 = [0,1]^{d_0}$. Now we study definite and indefinite integration of functions depending on a parameter $s \in Q_0$. Let $r_0, r \in \mathbb{N}_0$ and let $C^{r_0,r}(Q_0, Q)$ be the space of continuous functions $f : Q_0 \times Q \to \mathbb{K}$ having for $\alpha = (\alpha_0, \alpha_1)$, $\alpha_0 \in \mathbb{N}_0^{d_0}$, $\alpha_1 \in \mathbb{N}_0^d$ with $|\alpha_0| \le r_0$, $|\alpha_1| \le r$ continuous partial derivatives $\frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0} \partial t^{\alpha_1}}$, endowed with the norm

$$
\|f\|_{C^{r_0,r}(Q_0,Q)} = \max_{|\alpha_0| \le r_0, |\alpha_1| \le r} \sup_{s \in Q_0, t \in Q} \left| \frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0} \partial t^{\alpha_1}} \right|.
$$

Let furthermore $r_1 \in \mathbb{N}_0$ and put

$$
F = B_{C^{0,r}(Q_0,Q)} \cap B_{C^{r_0,r_1}(Q_0,Q)}.
$$

Note that for $r < r_1$ we have

$$
B_{C^{0,r}(Q_0,Q)} \cap B_{C^{r_0,r_1}(Q_0,Q)} = B_{C^{r_0,r_1}(Q_0,Q)} = B_{C^{0,r_1}(Q_0,Q)} \cap B_{C^{r_0,r_1}(Q_0,Q)},
$$

hence we can assume without loss of generality that $r \ge r_1$.

The definite parametric integration operator $\mathscr{S}_0 : C(Q_0 \times Q) \to C(Q_0)$ is given by

$$(\mathscr{S}_0 f)(s) = \int_Q f(s,t)dt \quad (s \in Q_0),$$

and the indefinite parametric integration operator $\mathscr{S}_1 : C(Q_0 \times Q) \to C(Q_0 \times Q)$ by

$$(\mathscr{S}_1 f)(s,t) = \int_{[0,t]} f(s,u)du \quad (s \in Q_0, t \in Q).$$

We consider standard information consisting of values of f, so the class of information functionals is $\Lambda = \{\delta_{s,t} : \ s \in Q_0, t \in Q\}$, where $\delta_{s,t}(f) = f(s,t)$. In the terminology of Section 2, the definite parametric integration problem is described by the tuple

$$\Pi_0 = (B_{C^{0,r}(Q_0,Q)} \cap B_{C^{r_0,r_1}(Q_0,Q)}, C(Q_0), \mathscr{S}_0, \mathbb{K}, \Lambda)$$

and the indefinite parametric integration problem by

$$\Pi_1 = (B_{C^{0,r}(Q_0,Q)} \cap B_{C^{r_0,r_1}(Q_0,Q)}, C(Q_0 \times Q), \mathscr{S}_1, \mathbb{K}, \Lambda).$$

The following theorem gives the complexity of definite and indefinite parametric integration. The case of definite parametric integration with $F = B_{C^r(Q_0 \times Q)}$ is already contained in [10], see also [2]. Definite parametric integration in Sobolev classes was considered in [6, 16]. The case of indefinite parametric integration with $F = B_{C^r(Q_0 \times Q)}$ was first studied in [2]. Below $\wedge$ and $\vee$ mean logical conjunction and disjunction, respectively.

Theorem 4.1. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $r \geq r_1$, $d, d_0 \in \mathbb{N}$, $\iota \in \{0,1\}$. Then the deterministic minimal errors satisfy*

$$\begin{array}{ll} e_n^{\det}(\mathscr{S}_\iota, F) \asymp n^{-\upsilon_1} & \text{if } \frac{r_0}{d_0} > \frac{r_1}{d} \\ n^{-\frac{r_0}{d_0}} \preceq e_n^{\det}(\mathscr{S}_\iota, F) \preceq n^{-\frac{r_0}{d_0}} (\log n)^{\frac{r_0}{d_0}+1} & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} > 0 \\ e_n^{\det}(\mathscr{S}_\iota, F) \asymp n^{-\frac{r_0}{d_0}} & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} = 0 \vee \frac{r_0}{d_0} < \frac{r_1}{d}, \end{array} \tag{11}$$

where

$$\upsilon_1 \ = \ \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + \frac{r}{d} - \frac{r_1}{d}} \frac{r}{d} \,. \tag{12}$$

Moreover, the randomized minimal errors fulfill

$$\begin{array}{ll} e_n^{\rm ran}(\mathscr{S}_\iota, F) \asymp n^{-\frac{r}{d}-\frac{1}{2}} & \text{if } \frac{r_0}{d_0} > \frac{r_1}{d} + \frac{1}{2} \wedge r = r_1 \\ e_n^{\rm ran}(\mathscr{S}_\iota, F) \asymp n^{-\upsilon_2} (\log n)^{\frac{1}{2}} & \text{if } \frac{r_0}{d_0} > \frac{r_1}{d} + \frac{1}{2} \wedge r > r_1 \\ n^{-\frac{r_0}{d_0}} (\log n)^{\frac{1}{2}} \preceq e_n^{\rm ran}(\mathscr{S}_\iota, F) \preceq n^{-\frac{r_0}{d_0}} (\log n)^{\frac{r_0}{d_0}+\frac{3}{2}} & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} + \frac{1}{2} \\ e_n^{\rm ran}(\mathscr{S}_\iota, F) \asymp n^{-\frac{r_0}{d_0}} (\log n)^{\frac{r_0}{d_0}-\frac{r_1}{d}} & \text{if } \frac{r_1}{d} < \frac{r_0}{d_0} < \frac{r_1}{d} + \frac{1}{2} \\ n^{-\frac{r_0}{d_0}} \preceq e_n^{\rm ran}(\mathscr{S}_\iota, F) \preceq n^{-\frac{r_0}{d_0}} (\log\log n)^{\frac{r_0}{d_0}+1} & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} > 0 \\ e_n^{\rm ran}(\mathscr{S}_\iota, F) \asymp n^{-\frac{r_0}{d_0}} & \text{if } \frac{r_0}{d_0} = \frac{r_1}{d} = 0 \vee \frac{r_0}{d_0} < \frac{r_1}{d}, \end{array} \tag{13}$$

with

$$\upsilon_2 = \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+\frac{r}{d}-\frac{r_1}{d}}\left(\frac{r}{d}+\frac{1}{2}\right). \tag{14}$$

For the proof we need some preparations and auxiliary statements. To connect parametric integration with Banach space valued integration considered in Section 3 we set $X = C(Q_0)$, $Y = C^{r_0}(Q_0)$, thus $C(Q_0 \times Q) = C(Q, X)$ and $\mathscr{S}_\iota = S_\iota^{C(Q_0)}$ ($\iota = 0, 1$). Moreover,

$$\begin{aligned} B_{C^{0,r}(Q_0,Q)} \cap B_{C^{r_0,r_1}(Q_0,Q)} &= B_{C^r(Q,C(Q_0))} \cap B_{C^{r_1}(Q,C^{r_0}(Q_0))} \\ &= B_{C^r(Q,X)} \cap B_{C^{r_1}(Q,Y)}. \end{aligned}$$

Let $r_2 = \max(r_0, 1)$ and define for $l \in \mathbb{N}_0$

$$T_l = P_{2^l}^{r_2,d_0} \in \mathscr{L}(C(Q_0)). \tag{15}$$

This way the algorithm $A_\omega^{(\iota)}$ defined in (7) becomes

$$A_\omega^{(\iota)} = P_{2^{l_0}}^{r_2,d_0} \otimes A_{n_{l_0},\omega}^{\iota,r} + \sum_{l=l_0+1}^{l_1} \left(P_{2^l}^{r_2,d_0} - P_{2^{l-1}}^{r_2,d_0}\right) \otimes A_{n_l,\omega}^{\iota,r_1}.$$

For $f \in C(Q_0 \times Q)$ this means

$$\begin{aligned} A_\omega^{(\iota)} f &= P_{2^{l_0}}^{r_2,d_0}\left(\left(A_{n_{l_0},\omega}^{\iota,r}(f_s)\right)_{s\in\Gamma_{r_2 2^{l_0}}^{d_0}}\right) \\ &\quad + \sum_{l=l_0+1}^{l_1} \left(P_{2^l}^{r_2,d_0} - P_{2^{l-1}}^{r_2,d_0}\right)\left(\left(A_{n_l,\omega}^{\iota,r_1}(f_s)\right)_{s\in\Gamma_{r_2 2^l}^{d_0}}\right), \end{aligned}$$

where for $s \in Q_0$ we used the notation $f_s = f(s, \cdot)$. Observe that

$$\operatorname{card}(A_\omega^{(\iota)}) \le c \sum_{l=l_0}^{l_1} n_l 2^{d_0 l} \quad (\omega \in \Omega). \tag{16}$$

First we estimate the error of $A_\omega^{(\iota)}$. Recall the notation $G_0(C(Q_0)) = C(Q_0)$ and $G_1(C(Q_0)) = C(Q_0 \times Q)$.

Proposition 4.2. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $r \ge r_1$, $\iota \in \{0, 1\}$. Then there are constants $c_1, c_2 > 0$ such that for all $l_0, l_1 \in \mathbb{N}_0$ with $l_0 \le l_1$ and for all $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ we have*

$$\begin{aligned} &\sup_{f\in F} \|\mathscr{S}_\iota f - A_\omega^{(\iota)} f\|_{G_\iota(C(Q_0))} \\ &\quad \le c_1 2^{-r_0 l_1} + c_1 n_{l_0}^{-r/d} + c_1 \sum_{l=l_0+1}^{l_1} 2^{-r_0 l} n_l^{-r_1/d} \quad (\omega \in \Omega) \end{aligned} \tag{17}$$

and for $l_0 \le l^* \le l_1$

$$\begin{aligned}
&\sup_{f\in F}\left(\mathbb{E}\,\|\mathscr{S}_\iota f - A_\omega^{(\iota)} f\|_{G_\iota(C(Q_0))}^2\right)^{1/2} \\
&\le\; c_2 2^{-r_0 l_1} + c_2 (l_0+1)^{1/2} n_{l_0}^{-r/d-1/2} \\
&\quad + c_2 \sum_{l=l_0+1}^{l^*} (l+1)^{1/2} 2^{-r_0 l} n_l^{-r_1/d-1/2} + c_2 \sum_{l=l^*+1}^{l_1} 2^{-r_0 l} n_l^{-r_1/d}. \qquad (18)
\end{aligned}$$

Proof. By (15) and (4),

$$\|T_l\|_{\mathscr{L}(C(Q_0))} \le c_1, \quad \|J - T_l J\|_{\mathscr{L}(C^{r_0}(Q_0),C(Q_0))} \le c_2 2^{-r_0 l}, \qquad (19)$$

where $J : C^{r_0}(Q_0) \to C(Q_0)$ is the embedding, and by (8) and (15),

$$X_l = P_{2^l}^{r_2,d_0}(C(Q_0)) = P_{2^l}^{r_2,d_0}(\ell_\infty(\Gamma_{r_2 2^l}^{d_0})).$$

Consequently, $X_{l-1} \subseteq X_l$ for $l \ge 1$, thus, $X_{l-1,l} \subseteq X_l$ and therefore

$$\tau_2(X_{l-1,l}) \le \tau_2(X_l). \qquad (20)$$

Moreover, it was observed in [2], proof of Proposition 4, that

$$\tau_2(X_l) \le c\tau_2\big(\ell_\infty\big(\Gamma_{r_2 2^l}^{d_0}\big)\big) \le c(l+1)^{1/2}. \qquad (21)$$

Now relations (17) and (18) are a direct consequence of Proposition 3.1 together with (19), (20), and (21). □

The following lemma contains the key estimates for the upper bound proof. It is formulated in a general way, which allows some shortcuts in the proof of Theorem 4.1. Moreover, it enables us to use these estimates directly for the analysis of parametric initial value problems in [3], where different but related smoothness classes are considered.

Let $\beta, \beta_0, \beta_1 \in \mathbb{R}$. Given $l_0, l^*, l_1 \in \mathbb{N}_0$ with $l_0 \le l^* \le l_1$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$, we define

$$M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) = 2^{-\beta_0 d_0 l_1} + n_{l_0}^{-\beta} + \sum_{l=l_0+1}^{l_1} 2^{-\beta_0 d_0 l} n_l^{-\beta_1} \qquad (22)$$

$$\begin{aligned}
E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big) &= 2^{-\beta_0 d_0 l_1} + (l_0+1)^{1/2} n_{l_0}^{-\beta} + \sum_{l=l_0+1}^{l^*} (l+1)^{1/2} 2^{-\beta_0 d_0 l} n_l^{-\beta_1} \\
&\quad + \sum_{l=l^*+1}^{l_1} 2^{-\beta_0 d_0 l} n_l^{-\beta_1+1/2}. \qquad (23)
\end{aligned}$$

Lemma 4.3. *Let $\beta, \beta_0, \beta_1 \in \mathbb{R}$ with $\beta_0 \geq 0$ and $\beta \geq \beta_1 \geq 0$. Then there are constants $c_{1-3} > 0$ such that for each $n \in \mathbb{N}$ with $n \geq 2$ there is a choice of $l_0, l_1 \in \mathbb{N}_0$, $l_0 \leq l_1$, and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ such that*

$$\sum_{l=l_0}^{l_1} n_l 2^{d_0 l} \leq c_1 n \tag{24}$$

and

$$\begin{aligned} & M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) \\ \leq \quad & c_2 \begin{cases} n^{-\upsilon} & \text{if} \quad \beta_0 > \beta_1 \\ n^{-\beta_0}(\log n)^{\beta_0+1} & \text{if} \quad \beta_0 = \beta_1 > 0 \\ n^{-\beta_0} & \text{if} \quad \beta_0 = \beta_1 = 0 \,\vee\, \beta_0 < \beta_1, \end{cases} \end{aligned} \tag{25}$$

where

$$\upsilon \quad = \quad \frac{\beta_0 \beta}{\beta_0 + \beta - \beta_1}. \tag{26}$$

Moreover, if $\beta_1 \geq 1/2$, then for each $n \in \mathbb{N}$ with $n > 2$ there is a choice of $l_0, l^*, l_1 \in \mathbb{N}_0$, $l_0 \leq l^* \leq l_1$, and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ satisfying (24) *and*

$$\begin{aligned} & E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big) \\ \leq \quad & c_3 \begin{cases} n^{-\beta} & \text{if} \quad \beta_0 > \beta_1 = \beta \\ n^{-\upsilon}(\log n)^{1/2} & \text{if} \quad \beta_0 > \beta_1 \,\wedge\, \beta > \beta_1 \\ n^{-\beta_0}(\log n)^{\beta_0+3/2} & \text{if} \quad \beta_0 = \beta_1 \\ n^{-\beta_0}(\log n)^{\beta_0-\beta_1+1/2} & \text{if} \quad \beta_1 - 1/2 < \beta_0 < \beta_1 \\ n^{-\beta_0}(\log\log n)^{\beta_0+1} & \text{if} \quad \beta_0 = \beta_1 - 1/2. \end{cases} \end{aligned} \tag{27}$$

Proof. In the case $\beta_0 = 0$ the statements trivially follow from (22) and (23) with $l_0 = l_1 = 0$ and $n_0 = 1$. Therefore, in the sequel we can assume $\beta_0 > 0$. Let $n \in \mathbb{N}$, $n \geq 2$, and put

$$l_1 = \left\lceil \frac{\log n}{d_0} \right\rceil, \quad l_0 = \left\lfloor \frac{\beta - \beta_1}{\beta_0 + \beta - \beta_1}\, l_1 \right\rfloor \tag{28}$$

(recall that log always means $\log_2$). We note that (28) implies

$$l_1 - l_0 \geq \frac{\beta_0 l_1}{\beta_0 + \beta - \beta_1}, \tag{29}$$

hence

$$(\beta - \beta_1)(l_1 - l_0) \geq \frac{(\beta - \beta_1)\beta_0 l_1}{\beta_0 + \beta - \beta_1} \geq \beta_0 l_0,$$

and thus

$$\beta(l_1 - l_0) \geq \beta_0 l_0 + \beta_1 (l_1 - l_0). \tag{30}$$

Let $\sigma \in \{0,1\}$, $\delta_0, \delta_1 \geq 0$ to be fixed later on and set

$$n_{l_0} = 2^{d_0(l_1-l_0)}, \tag{31}$$

$$n_l = \left\lceil (l_1+1)^{-\sigma} 2^{d_0(l_1-l)-\delta_0(l-l_0)-\delta_1(l_1-l)} \right\rceil \quad (l = l_0+1, \dots, l_1). \tag{32}$$

This gives

$$\sum_{l=l_0}^{l_1} n_l 2^{d_0 l} \leq c2^{d_0 l_1} + (l_1+1)^{-\sigma} \sum_{l=l_0+1}^{l_1} 2^{d_0 l_1 - \delta_0(l-l_0)-\delta_1(l_1-l)} \leq cn, \tag{33}$$

provided $\delta_0 > 0$ or $\delta_1 > 0$ or $\sigma = 1$. By (30) and (31) we have

$$n_{l_0}^{-\beta} = 2^{-\beta d_0(l_1-l_0)} \leq 2^{-\beta_0 d_0 l_0 - \beta_1 d_0(l_1-l_0)} \leq 2^{-\beta_0 d_0 l_0 - \beta_1 d_0(l_1-l_0)+\beta_1\delta_1(l_1-l_0)} \tag{34}$$

and, using (32), for $l_0 < l \leq l_1$

$$2^{-\beta_0 d_0 l} n_l^{-\beta_1} \leq (l_1+1)^{\sigma\beta_1} 2^{-\beta_0 d_0 l - \beta_1 d_0(l_1-l)+\beta_1\delta_0(l-l_0)+\beta_1\delta_1(l_1-l)}. \tag{35}$$

Furthermore,

$$\begin{aligned} &-\beta_0 d_0 l - \beta_1 d_0(l_1-l) + \beta_1\delta_0(l-l_0) + \beta_1\delta_1(l_1-l) \\ &= -\beta_0 d_0 l_0 - (\beta_0 d_0 - \beta_1\delta_0)(l-l_0) - \beta_1(d_0-\delta_1)(l_1-l) \quad (l_0 \leq l \leq l_1). \end{aligned} \tag{36}$$

By (22) and (34–36),

$$\begin{aligned} &M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) \\ &\leq 2^{-\beta_0 d_0 l_1} + (l_1+1)^{\sigma\beta_1} \sum_{l=l_0}^{l_1} 2^{-\beta_0 d_0 l_0 - (\beta_0 d_0 - \beta_1\delta_0)(l-l_0) - \beta_1(d_0-\delta_1)(l_1-l)}. \end{aligned} \tag{37}$$

If $\beta_0 > \beta_1$, we set $\sigma = \delta_1 = 0$ and choose $\delta_0 > 0$ in such a way that $\beta_0 d_0 - \beta_1\delta_0 > \beta_1 d_0$. From (37) we obtain

$$\begin{aligned} M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) &\leq 2^{-\beta_0 d_0 l_1} + \sum_{l=l_0}^{l_1} 2^{-\beta_0 d_0 l_0 - (\beta_0 d_0 - \beta_1\delta_0)(l-l_0) - \beta_1 d_0(l_1-l)} \\ &\leq 2^{-\beta_0 d_0 l_1} + c2^{-\beta_0 d_0 l_0 - \beta_1 d_0(l_1-l_0)}. \end{aligned} \tag{38}$$

Note that by (26), (28), and (29)

$$\begin{aligned} \beta_0 l_0 + \beta_1(l_1-l_0) &\geq \frac{\beta_0(\beta-\beta_1)l_1}{\beta_0+\beta-\beta_1} - \beta_0 + \frac{\beta_1\beta_0 l_1}{\beta_0+\beta-\beta_1} \\ &= \frac{\beta_0\beta l_1}{\beta_0+\beta-\beta_1} - \beta_0 = \upsilon l_1 - \beta_0 \end{aligned} \tag{39}$$

and, since $\beta_0 > \beta_1$,

$$\upsilon = \frac{\beta_0\beta}{\beta_0+\beta-\beta_1} < \beta_0. \tag{40}$$

It follows from (28) and (38–40) that

$$M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) \;\le\; 2^{-\beta_0 d_0 l_1} + c2^{-v d_0 l_1} \le c2^{-v d_0 l_1} \le cn^{-v}.$$

This together with (33) proves (25) for $\beta_0 > \beta_1$.

If $\beta_0 = \beta_1 > 0$, we set $\sigma = 1$, $\delta_0 = \delta_1 = 0$, and get from (28) and (37)

$$\begin{aligned} M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) &\le 2^{-\beta_0 d_0 l_1} + (l_1+1)^{\beta_0} \sum_{l=l_0}^{l_1} 2^{-\beta_0 d_0 l_0 - \beta_0 d_0 (l-l_0) - \beta_0 d_0 (l_1 - l)} \\ &\le c(l_1+1)^{\beta_0+1} 2^{-\beta_0 d_0 l_1} \le cn^{-\beta_0} (\log n)^{\beta_0+1}. \end{aligned}$$

Combining this with (33) gives the respective estimate of (25).

Since we assumed $\beta_0 > 0$, it remains to consider the case $\beta_0 < \beta_1$, where we set $\sigma = \delta_0 = 0$ and choose $\delta_1 > 0$ in such a way that $\beta_1(d_0 - \delta_1) > \beta_0 d_0$. By (28) and (37),

$$\begin{aligned} M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big) &\le 2^{-\beta_0 d_0 l_1} + \sum_{l=l_0}^{l_1} 2^{-\beta_0 d_0 l_0 - \beta_0 d_0 (l-l_0) - \beta_1 (d_0 - \delta_1)(l_1 - l)} \\ &\le 2^{-\beta_0 d_0 l_1} + c2^{-\beta_0 d_0 l_0 - \beta_0 d_0 (l_1 - l_0)} \le cn^{-\beta_0}. \end{aligned}$$

This together with (33) completes the proof of (25).

Now we turn to the proof of (27) and assume that $\beta_1 \ge 1/2$. If $\beta_0 > \beta_1 = \beta$, then we set $l^* = l_1$, $\sigma = \delta_1 = 0$ and choose $\delta_0 > 0$ satisfying $\beta_0 d_0 - \beta_1 \delta_0 > \beta_1 d_0$. It follows from (28) that $l_0 = 0$. Then (23), (34), and (35) give

$$\begin{aligned} &E\big(l_0, l_1, l_1, (n_l)_{l=l_0}^{l_1}\big) \\ &\le 2^{-\beta_0 d_0 l_1} + 2^{-\beta_1 d_0 l_1} + \sum_{l=1}^{l_1} (l+1)^{1/2} 2^{-\beta_0 d_0 l - \beta_1 d_0 (l_1 - l) + \beta_1 \delta_0 l} \\ &\le 2^{-\beta_0 d_0 l_1} + \sum_{l=0}^{l_1} (l+1)^{1/2} 2^{-(\beta_0 d_0 - \beta_1 \delta_0) l - \beta_1 d_0 (l_1 - l)} \\ &\le 2^{-\beta_0 d_0 l_1} + c2^{-\beta d_0 l_1} \le cn^{-\beta}, \end{aligned}$$

which together with (33) proves the first case of (27).

If $(\beta_0 > \beta_1 \wedge \beta > \beta_1)$ or $\beta_0 = \beta_1$, we choose $l^* = l_1$, get from (22–23)

$$E\big(l_0, l_1, l_1, (n_l)_{l=l_0}^{l_1}\big) \;\le\; (l_1+1)^{1/2} M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big),$$

and the desired results follow from (28) and the respective cases of (25).

It remains to consider the case

$$\beta_1 - 1/2 \le \beta_0 < \beta_1. \tag{41}$$

Here we make another choice of the parameters $(n_l)_{l=l_0}^{l_1}$ (while l_0 and l_1 remain the same, given by (28)). Let $\sigma \in \{0,1\}$, $\delta_1, \delta_2 \ge 0$, and $l^* \in \mathbb{N}_0$ with $l_0 \le l^* \le l_1$ to be fixed later on and set

$$n_{l_0} = 2^{d_0(l_1-l_0)}, \tag{42}$$
$$n_l = \left\lceil 2^{d_0(l_1-l)-\delta_1(l^*-l)} \right\rceil \quad (l = l_0+1,\dots,l^*), \tag{43}$$
$$n_l = \left\lceil (l_1-l^*+1)^{-\sigma} 2^{d_0(l_1-l)-\delta_2(l-l^*)} \right\rceil \quad (l = l^*+1,\dots,l_1). \tag{44}$$

In the sequel we need the following estimate, which results from (28) and (42–44).

$$\begin{aligned}\sum_{l=l_0}^{l_1} n_l 2^{d_0 l} &\le c2^{d_0 l_1} + \sum_{l=l_0+1}^{l^*} 2^{d_0 l_1 - \delta_1(l^*-l)} + (l_1-l^*+1)^{-\sigma} \sum_{l=l^*+1}^{l_1} 2^{d_0 l_1 - \delta_2(l-l^*)} \\ &\le cn \end{aligned} \tag{45}$$

whenever $(\delta_1 > 0 \wedge \delta_2 > 0)$ or $(\sigma = 1 \wedge \delta_1 > 0)$. Using (30) and (42), we obtain

$$n_{l_0}^{-\beta} = 2^{-\beta d_0(l_1-l_0)} \le 2^{-\beta_0 d_0 l_0 - \beta_1 d_0(l_1-l_0)} \le 2^{-\beta_0 d_0 l - \beta_1 d_0(l_1-l_0) + \beta_1\delta_1(l^*-l_0)}. \tag{46}$$

Denote $\beta_2 = \beta_1 - 1/2$. From (43–44) we get

$$2^{-\beta_0 d_0 l} n_l^{-\beta_1} \le 2^{-\beta_0 d_0 l - \beta_1 d_0(l_1-l) + \beta_1\delta_1(l^*-l)} \quad (l_0 < l \le l^*) \tag{47}$$
$$2^{-\beta_0 d_0 l} n_l^{-\beta_2} \le (l_1-l^*+1)^{\sigma\beta_2} 2^{-\beta_0 d_0 l - \beta_2 d_0(l_1-l) + \beta_2\delta_2(l-l^*)} \quad (l^* < l \le l_1). \tag{48}$$

Moreover, we have for $l_0 \le l \le l^*$

$$\begin{aligned}&-\beta_0 d_0 l - \beta_1 d_0(l_1-l) + \beta_1\delta_1(l^*-l) \\ &= -\beta_0 d_0 l_0 - \beta_1 d_0(l_1-l^*) - \beta_0 d_0(l-l_0) - \beta_1(d_0-\delta_1)(l^*-l)\end{aligned} \tag{49}$$

and for $l^*+1 \le l \le l_1$

$$\begin{aligned}&-\beta_0 d_0 l - \beta_2 d_0(l_1-l) + \beta_2\delta_2(l-l^*) \\ &= -\beta_0 d_0 l^* - \beta_2 d_0(l_1-l) - (\beta_0 d_0 - \beta_2\delta_2)(l-l^*).\end{aligned} \tag{50}$$

Now (23) and (46–50) imply

$$E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big) \le 2^{-\beta_0 d_0 l_1} + E_1 + E_2, \tag{51}$$

where

$$E_1 = \sum_{l=l_0}^{l^*} (l+1)^{1/2} 2^{-\beta_0 d_0 l_0 - \beta_1 d_0(l_1-l^*) - \beta_0 d_0(l-l_0) - \beta_1(d_0-\delta_1)(l^*-l)} \tag{52}$$

$$E_2 = (l_1-l^*+1)^{\sigma\beta_2} \sum_{l=l^*+1}^{l_1} 2^{-\beta_0 d_0 l^* - \beta_2 d_0(l_1-l) - (\beta_0 d_0 - \beta_2\delta_2)(l-l^*)}. \tag{53}$$

Put

$$l^* = l_1 - \left\lceil \frac{\log(l_1+1)}{d_0} \right\rceil \tag{54}$$

and observe that the assumption $\beta_0 > 0$, (28), and (54) imply that there is a constant $c_0 \in \mathbb{N}$ such that for $n \ge c_0$

$$l_0 < l^* \le l_1. \tag{55}$$

Since for $n < c_0$ the estimate (27) trivially follows from (51–53) by a suitable choice of the constant, we can assume $n \ge c_0$, and thus (55). Using (41), we choose $\delta_1 > 0$ in such a way that $\beta_0 d_0 < \beta_1(d_0 - \delta_1)$. Then by (52), (54), and (28)

$$\begin{aligned} E_1 &\le c(l_1+1)^{1/2} 2^{-\beta_0 d_0 l_0 - \beta_1 d_0 (l_1 - l^*) - \beta_0 d_0 (l^* - l_0)} \\ &= c(l_1+1)^{1/2} 2^{-\beta_0 d_0 l_1 + (\beta_0 d_0 - \beta_1 d_0)(l_1 - l^*)} \le c(l_1+1)^{\beta_0 - \beta_1 + 1/2} 2^{-\beta_0 d_0 l_1} \\ &\le c n^{-\beta_0} (\log n)^{\beta_0 - \beta_1 + 1/2}. \end{aligned} \tag{56}$$

Now we deal with E_2 and distinguish between two subcases of (41). If $\beta_1 - 1/2 < \beta_0$, we set $\sigma = 0$ and choose $\delta_2 > 0$ in such a way that $\beta_2 d_0 < \beta_0 d_0 - \beta_2 \delta_2$ (recall that $\beta_2 = \beta_1 - 1/2$). Then, using (28), (53), and (54),

$$\begin{aligned} E_2 &\le c 2^{-\beta_0 d_0 l^* - \beta_2 d_0 (l_1 - l^*)} = c 2^{-\beta_0 d_0 l_1 + (\beta_0 d_0 - \beta_2 d_0)(l_1 - l^*)} \\ &\le c 2^{-\beta_0 d_0 l_1} (l_1+1)^{\beta_0 - \beta_1 + 1/2} \le c n^{-\beta_0} (\log n)^{\beta_0 - \beta_1 + 1/2}. \end{aligned} \tag{57}$$

Combining (51) and (56–57), and taking into account (45), we obtain the fourth case of (27). If $\beta_1 - 1/2 = \beta_0$ (and thus, $\beta_2 = \beta_0$), we set $\sigma = 1$ and $\delta_2 = 0$. Here we have

$$E_2 \le c(l_1 - l^* + 1)^{\beta_2 + 1} 2^{-\beta_0 d_0 l_1} \le c n^{-\beta_0} (\log\log n)^{\beta_0 + 1}. \tag{58}$$

The last case of (27) is now a consequence of (51), (56), (58), and (45).

□

Proof of the upper bounds in Theorem 4.1.

We derive the upper bounds in (11) and (13) from (17), (18) of Proposition 4.2 and Lemma 4.3. To deal with (11) we set

$$\beta = \frac{r}{d}, \quad \beta_0 = \frac{r_0}{d_0}, \quad \beta_1 = \frac{r_1}{d}, \tag{59}$$

which together with (26) and (12) gives for $r_0/d_0 > r_1/d$

$$\upsilon = \frac{\beta_0 \beta}{\beta_0 + \beta - \beta_1} = \frac{\frac{r_0}{d_0} \cdot \frac{r}{d}}{\frac{r_0}{d_0} + \frac{r}{d} - \frac{r_1}{d}} = \upsilon_1. \tag{60}$$

Furthermore, note that (17) and (22) imply

$$\sup_{f \in F} \| \mathscr{S}_\iota f - A_\omega^{(\iota)} f \|_{G_\iota(C(Q_0))} \le c\, M\big(l_0, l_1, (n_l)_{l=l_0}^{l_1}\big). \tag{61}$$

Now the upper bounds in (11) follows from (16), (24–25), and (59–61). Finally we consider (13) and put

$$\beta = \frac{r}{d} + \frac{1}{2}, \quad \beta_0 = \frac{r_0}{d_0}, \quad \beta_1 = \frac{r_1}{d} + \frac{1}{2}, \tag{62}$$

which gives for $r_0/d_0 > r_1/d + 1/2$

$$\upsilon = \frac{\beta_0\beta}{\beta_0 + \beta - \beta_1} = \frac{\frac{r_0}{d_0}\left(\frac{r}{d} + \frac{1}{2}\right)}{\frac{r_0}{d_0} + \frac{r}{d} - \frac{r_1}{d}} = \upsilon_2. \tag{63}$$

We conclude from (18) and (23) that for any l^* with $l_0 \le l^* \le l_1$

$$\sup_{f\in F}\left(\mathbb{E}\,\|\mathscr{S}_\iota f - A_\omega^{(\iota)} f\|^2_{G_\iota(C(Q_0))}\right)^{1/2} \le c\,E\big(l_0, l^*, l_1, (n_l)_{l=l_0}^{l_1}\big). \tag{64}$$

With this, the upper estimates in (13) are a consequence of (16), (24), (27), and (62–64), except for the last case of (13), which follows directly from the respective case of the deterministic setting (11).

□

For the proof of the lower bounds we let $\varphi_0 \not\equiv 0$ be a C^∞ function on $\mathbb{R}^{d_0}$ with support in Q_0 and φ_1 a C^∞ function on $\mathbb{R}^d$ with support in Q and $\int_Q \varphi_1(t)dt \neq 0$. Let $m_0, m_1 \in \mathbb{N}$, let $Q_{0,i}$ ($i = 1, \dots, m_0^{d_0}$) be the subdivision of Q_0 into $m_0^{d_0}$ cubes of disjoint interior of sidelength m_0^{-1} and let $Q_{1,j}$ ($j = 1, \dots, m_1^d$) be the respective subdivision of Q. Let s_i, respectively t_j, be the point in $Q_{0,i}$, respectively $Q_{1,j}$, with minimal coordinates. Define for $s \in Q_0$, $t \in Q$, $i = 1, \dots, m_0^{d_0}$, $j = 1, \dots, m_1^d$

$$\varphi_{0,i}(s) = \varphi_0(m_0(s - s_i)), \quad \varphi_{1,j}(t) = \varphi_1(m_1(t - t_j)),$$

and

$$\psi_{ij}(s,t) = \varphi_{0,i}(s)\varphi_{1,j}(t).$$

Denote

$$\Psi^0_{m_0,m_1} = \left\{\sum_{i=1}^{m_0^{d_0}}\sum_{j=1}^{m_1^d} \delta_{ij}\psi_{ij} :\ \delta_{ij} \in [-1,1],\ i = 1, \dots, m_0^{d_0}, j = 1, \dots, m_1^d\right\}.$$

Lemma 4.4. *Let $\iota \in \{0,1\}$. Then there are constants $c_1, c_2 > 0$ such that for all $m_0, m_1, n \in \mathbb{N}$ with*

$$m_0^{d_0} m_1^d \ge 4n \tag{65}$$

we have

$$e_n^{\rm det}(\mathscr{S}_\iota, \Psi^0_{m_0,m_1}) \ \ge\ c_1 \tag{66}$$

$$e_n^{\rm ran}(\mathscr{S}_\iota, \Psi^0_{m_0,m_1}) \ \ge\ c_2 m_1^{-d/2} \min\left(m_1^d, \log(m_0+1)\right)^{1/2}. \tag{67}$$

Proof. Define $S_2 : C(Q_0 \times Q) \to \mathbb{K}$ to be the integration operator

$$S_2 f = \int_{Q_0 \times Q} f(s,t) ds dt \quad (f \in C(Q_0 \times Q)).$$

Then

$$e_n^{\text{det}}(\mathscr{S}_\iota, \Psi^0_{m_0,m_1}) \geq e_n^{\text{det}}(S_2, \Psi^0_{m_0,m_1}).$$

Moreover, standard results from [15], Ch. 4.5, give

$$e_n^{\text{det}}(S_2, \Psi^0_{m_0,m_1}) \geq c,$$

which implies (66). The bound (67) was shown for $\mathscr{S}_0$ in [10], see the proof of Proposition 5.1 and, in particular, relations (48) and (58), combined with Lemma 5 there. Since $\mathscr{S}_0$ is a particular case of $\mathscr{S}_1$ (in other words, $\mathscr{S}_0$ reduces to $\mathscr{S}_1$, see [8] for a formal definition), the lower bound also holds for $\mathscr{S}_1$.

□

We proceed by providing another technical estimate, which is again formulated in somewhat general terms, also in view of further application in [3]. For $\gamma, \gamma_0, \gamma_1 \in \mathbb{R}$ we define

$$\Psi^{\gamma,\gamma_0,\gamma_1}_{m_0,m_1} = \min\left(m_1^{-\gamma}, m_0^{-\gamma_0} m_1^{-\gamma_1}\right) \Psi^0_{m_0,m_1}. \tag{68}$$

Lemma 4.5. *Let $\iota \in \{0,1\}$ and $\gamma, \gamma_0, \gamma_1 \in \mathbb{R}$ with $\gamma_0 \geq 0$ and $\gamma \geq \gamma_1 \geq 0$. Then there are constants $c_1, c_2 > 0$ such that for each $n \in \mathbb{N}$ with $n \geq 2$ there is a choice of $m_0, m_1 \in \mathbb{N}_0$ fulfilling (65) and*

$$e_n^{\text{det}}(\mathscr{S}_\iota, \Psi^{\gamma,\gamma_0,\gamma_1}_{m_0,m_1}) \quad \geq \quad c_1 \begin{cases} n^{-\upsilon_3} & \text{if} \quad \gamma_0/d_0 > \gamma_1/d \\ n^{-\gamma_0/d_0} & \text{if} \quad \gamma_0/d_0 \leq \gamma_1/d, \end{cases}$$

where υ_3 is defined by

$$\upsilon_3 \quad = \quad \frac{\gamma_0 \gamma}{\gamma_0 d + (\gamma - \gamma_1) d_0}. \tag{69}$$

Furthermore, for each $n \in \mathbb{N}$ with $n > 2$ there is a choice of $m_0, m_1 \in \mathbb{N}_0$ such that (65) holds and

$$\begin{aligned} & e_n^{\text{ran}}(\mathscr{S}_\iota, \Psi^{\gamma,\gamma_0,\gamma_1}_{m_0,m_1}) \\ \geq \quad & c_2 \begin{cases} n^{-\gamma/d - 1/2} & \text{if} \quad \gamma_0/d_0 > \gamma_1/d + 1/2 \wedge \gamma = \gamma_1 \\ n^{-\upsilon_4} (\log n)^{1/2} & \text{if} \quad \gamma_0/d_0 > \gamma_1/d + 1/2 \wedge \gamma > \gamma_1 \\ n^{-\gamma_0/d_0} (\log n)^{\gamma_0/d_0 - \gamma_1/d} & \text{if} \quad \gamma_1/d < \gamma_0/d_0 \leq \gamma_1/d + 1/2 \\ n^{-\gamma_0/d_0} & \text{if} \quad \gamma_0/d_0 \leq \gamma_1/d. \end{cases} \end{aligned}$$

with

$$\upsilon_4 \quad = \quad \frac{\gamma_0(\gamma + d/2)}{\gamma_0 d + (\gamma - \gamma_1) d_0}. \tag{70}$$

Proof. Let $n \in \mathbb{N}$, $n \geq 2$. We start with the deterministic setting. First consider the case $\gamma_0/d_0 > \gamma_1/d$ and put

$$m_0 = 2\left\lceil n^{\frac{\gamma-\gamma_1}{\gamma_0 d+(\gamma-\gamma_1)d_0}}\right\rceil, \quad m_1 = 2\left\lceil n^{\frac{\gamma_0}{\gamma_0 d+(\gamma-\gamma_1)d_0}}\right\rceil. \tag{71}$$

It follows that $m_0^{d_0} m_1^d \geq 4n$ and

$$\min\left(m_1^{-\gamma}, m_0^{-\gamma_0} m_1^{-\gamma_1}\right) \geq cn^{-\frac{\gamma_0\gamma}{\gamma_0 d+(\gamma-\gamma_1)d_0}} = cn^{-\upsilon_3}. \tag{72}$$

This together with (66) and (68) yields

$$e_n^{\mathrm{det}}(\mathscr{S}_\iota, \Psi_{m_0,m_1}^{\gamma,\gamma_0,\gamma_1}) \geq cn^{-\upsilon_3}.$$

Next suppose $\gamma_0/d_0 \leq \gamma_1/d$. Here we put

$$m_0 = 4\left\lceil n^{1/d_0}\right\rceil, \quad m_1 = 1. \tag{73}$$

Clearly, $m_0^{d_0} m_1^d \geq 4n$ and

$$\min\left(m_1^{-\gamma}, m_0^{-\gamma_0} m_1^{-\gamma_1}\right) \geq cn^{-\gamma_0/d_0}, \tag{74}$$

therefore, by (66) and (68)

$$e_n^{\mathrm{det}}(\mathscr{S}_\iota, \Psi_{m_0,m_1}^{\gamma,\gamma_0,\gamma_1}) \geq cn^{-\gamma_0/d_0}.$$

Now we turn to the randomized setting. First we consider the case $\gamma_0/d_0 > \gamma_1/d + 1/2$. Define m_0, m_1 as in (71). Then we have

$$\begin{aligned} \min(m_1^d, \log(m_0+1))^{1/2} &\geq \begin{cases} c & \text{if} \quad \gamma = \gamma_1 \\ c(\log n)^{1/2} & \text{if} \quad \gamma > \gamma_1 \end{cases} \\ m_1^{-d/2} &\geq cn^{-\frac{\gamma_0 d/2}{\gamma_0 d+(\gamma-\gamma_1)d_0}}, \end{aligned}$$

thus, using (67) and (72),

$$\begin{aligned} &e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi_{m_0,m_1}^{\gamma,\gamma_0,\gamma_1}) \\ &\geq \begin{cases} cn^{-\frac{\gamma_0(\gamma+d/2)}{\gamma_0 d+(\gamma-\gamma_1)d_0}} = cn^{-\frac{\gamma}{d}-\frac{1}{2}} & \text{if} \quad \gamma = \gamma_1 \\ cn^{-\frac{\gamma_0(\gamma+d/2)}{\gamma_0 d+(\gamma-\gamma_1)d_0}}(\log n)^{1/2} = cn^{-\upsilon_4}(\log n)^{1/2} & \text{if} \quad \gamma > \gamma_1. \end{cases} \end{aligned}$$

Next we consider the case $\gamma_1/d < \gamma_0/d_0 \leq \gamma_1/d + 1/2$. Here we put

$$m_0 = 2\left\lceil\left(\frac{n}{\log n}\right)^{1/d_0}\right\rceil, \quad m_1 = 2\left\lceil(\log n)^{1/d}\right\rceil,$$

which again implies $m_0^{d_0} m_1^d \geq 4n$. Furthermore, we have

$$\begin{aligned} \min(m_1^d, \log(m_0+1))^{1/2} &\geq c(\log n)^{1/2}, \\ m_1^{-d/2} &\geq c(\log n)^{-1/2} \\ \min\left(m_1^{-\gamma}, m_0^{-\gamma_0} m_1^{-\gamma_1}\right) &\geq cn^{-\gamma_0/d_0}(\log n)^{\gamma_0/d_0-\gamma_1/d}. \end{aligned}$$

Combining this with (67) gives

$$e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi_{m_0,m_1}^{\gamma,\gamma_0,\gamma_1}) \geq n^{-\gamma_0/d_0}(\log n)^{\gamma_0/d_0-\gamma_1/d}.$$

Finally, let $\gamma_0/d_0 \leq \gamma_1/d$. Here we use the choice (73) and have

$$\min(m_1^d, \log(m_0+1))^{1/2} \geq c.$$

This together with (74) yields

$$e_n^{\mathrm{ran}}(\mathscr{S}_\iota, \Psi_{m_0,m_1}^{\gamma,\gamma_0,\gamma_1}) \geq n^{-\gamma_0/d_0}.$$

□

Proof of the lower bounds in Theorem 4.1.
Observe that there is a constant $c > 0$ such that for all $m_0, m_1 \in \mathbb{N}$

$$c \min\left(m_1^{-r}, m_0^{-r_0} m_1^{-r_1}\right) \Psi_{m_0,m_1}^0 \subset F.$$

Consequently, by (68),

$$e_n^{\mathrm{set}}(\mathscr{S}_\iota, F) \geq c\, e_n^{\mathrm{set}}(\mathscr{S}_\iota, \Psi_{m_0,m_1}^{r,r_0,r_1}) \quad (m_0, m_1 \in \mathbb{N}),$$

where set $\in \{\mathrm{det}, \mathrm{ran}\}$. Setting $\gamma = r$, $\gamma_0 = r_0$, $\gamma_1 = r_1$, we have by (12) and (69), for $r_0/d_0 > r_1/d$

$$\upsilon_3 = \frac{r_0 r}{r_0 d + (r - r_1) d_0} = \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + \frac{r}{d} - \frac{r_1}{d}} \, \frac{r}{d} = \upsilon_1,$$

and, by (14) and (70), for $r_0/d_0 > r_1/d + 1/2$

$$\upsilon_4 = \frac{r_0(r + d/2)}{r_0 d + (r - r_1) d_0} = \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0} + \frac{r}{d} - \frac{r_1}{d}} \left(\frac{r}{d} + \frac{1}{2}\right) = \upsilon_2.$$

Now Lemma 4.5 yields the lower bounds.

□

5 Some particular classes and comments

If $r_1 = r$, then $F = B_{C^{r_0,r}(Q_0,Q)}$, which is a class of dominating mixed smoothness, more precisely, the smoothness with respect to the parameter variables s and the smoothness with respect to the variables t are combined in such a way.

Corollary 5.1. *Let $r_0, r \in \mathbb{N}_0$, $r_1 = r$, $d, d_0 \in \mathbb{N}$, $\iota \in \{0,1\}$. Then*

$$\begin{aligned} e_n^{\rm det}(\mathscr{S}_\iota, F) &\asymp_{\log} n^{-\min\left(\frac{r}{d}, \frac{r_0}{d_0}\right)} \\ e_n^{\rm ran}(\mathscr{S}_\iota, F) &\asymp_{\log} n^{-\min\left(\frac{r}{d}+\frac{1}{2}, \frac{r_0}{d_0}\right)}. \end{aligned}$$

Let us compare the order of the deterministic and randomized minimal errors neglecting logarithmic factors. If the smoothness r_0 with respect to the parameter satisfies $r_0/d_0 \geq r/d + 1/2$, then the order of $e_n^{\rm ran}(\mathscr{S}_\iota, F)$ is the same as that of the randomized minimal errors for (nonparametric) integration of functions from $C^r(Q)$ and is by $n^{-1/2}$ faster than parametric integration in the deterministic setting. If $r/d < r_0/d_0 < r/d + 1/2$, the randomized rate is still superior, but the gap becomes smaller and reaches zero when $r_0/d_0 \leq r/d$.

Next consider the case $r_1 = 0$. This leads to the class

$$C^{(r_0,0)\wedge(0,r)}(Q_0,Q) := C^{r_0,0}(Q_0,Q) \cap C^{0,r}(Q_0,Q)$$

of continuous functions $f : Q_0 \times Q \to \mathbb{K}$ having for $\alpha_0 \in \mathbb{N}_0^{d_0}$ with $|\alpha_0| \leq r_0$ and for $\alpha_1 \in \mathbb{N}_0^d$ with $|\alpha_1| \leq r$ continuous partial derivatives $\frac{\partial^{|\alpha_0|} f(s,t)}{\partial s^{\alpha_0}}$ and $\frac{\partial^{|\alpha_1|} f(s,t)}{\partial t^{\alpha_1}}$, endowed with the norm

$$\begin{aligned} &\|f\|_{C^{(r_0,0)\wedge(0,r)}(Q_0,Q)} \\ &= \max\left(\|f\|_{C^{r_0,0}(Q_0,Q)}, \|f\|_{C^{0,r}(Q_0,Q)}\right) \\ &= \max\left(\max_{|\alpha_0|\leq r_0} \sup_{s\in Q_0, t\in Q} \left|\frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0}}\right|, \max_{|\alpha_1|\leq r} \sup_{s\in Q_0, t\in Q} \left|\frac{\partial^{|\alpha|} f(s,t)}{\partial t^{\alpha_1}}\right|\right). \end{aligned}$$

Thus, here we consider separate differentiability with respect to the s- and t-variables. Before we state the result, we want to mention a closely related subclass. Let $C^{[r_0,r]}(Q_0,Q)$ denote the class of continuous functions possessing continuous partial derivatives $\frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0} \partial t^{\alpha_1}}$ for all $\alpha_0 \in \mathbb{N}_0^{d_0}$, $\alpha_1 \in \mathbb{N}_0^d$ satisfying $\frac{|\alpha_0|}{r_0} + \frac{|\alpha_1|}{r} \leq 1$ (with the convention $\frac{0}{0} = 0$ and $\frac{c}{0} = +\infty$ for $c > 0$), equipped with the norm

$$\|f\|_{C^{[r_0,r]}(Q_0,Q)} = \max_{\frac{|\alpha_0|}{r_0} + \frac{|\alpha_1|}{r} \leq 1} \sup_{s\in Q_0, t\in Q} \left|\frac{\partial^{|\alpha|} f(s,t)}{\partial s^{\alpha_0} \partial t^{\alpha_1}}\right|.$$

For $r_0 = r$ this is just the class $C^r(Q_0,Q)$. Clearly, we have

$$C^{[r_0,r]}(Q_0,Q) \subseteq C^{(r_0,0)\wedge(0,r)}(Q_0,Q) \tag{75}$$

and

$$\|f\|_{C^{[r_0,r]}(Q_0,Q)} \ge \|f\|_{C^{(r_0,0)\wedge(0,r)}(Q_0,Q)}.$$

In general, the inclusion in (75) is strict, see [13, 1].

Corollary 5.2. *Let $r_0, r \in \mathbb{N}_0$, $d, d_0 \in \mathbb{N}$, $\iota \in \{0,1\}$ and let F_1 be any set with*

$$B_{C^{[r_0,r]}(Q_0,Q)} \subseteq F_1 \subseteq B_{C^{(r_0,0)\wedge(0,r)}(Q_0,Q)}. \tag{76}$$

Then

$$\begin{aligned} e_n^{\rm det}(\mathscr{S}_\iota, F_1) &\asymp_{\log} n^{-\upsilon_5} \\ e_n^{\rm ran}(\mathscr{S}_\iota, F_1) &\asymp_{\log} n^{-\upsilon_6}, \end{aligned}$$

with

$$\upsilon_5 = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+\frac{r}{d}}\, \frac{r}{d} & \text{if} \quad r_0 > 0 \\ 0 & \text{if} \quad r_0 = 0 \end{cases} \tag{77}$$

$$\upsilon_6 = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+\frac{r}{d},}\left(\frac{r}{d}+\frac{1}{2}\right) & \text{if} \quad \frac{r_0}{d_0} > \frac{1}{2} \\ \frac{r_0}{d_0} & \text{if} \quad \frac{r_0}{d_0} \le \frac{1}{2}. \end{cases} \tag{78}$$

Proof. The upper bounds follow from (76) and Theorem 4.1. For the proof of the lower bounds we observe that there is a constant $c > 0$ such that for all $m_0, m_1 \in \mathbb{N}$, $\psi \in \Psi^0_{m_0,m_1}$

$$\begin{aligned} &\|\psi\|_{C^{[r_0,r]}(Q_0\times Q)} \\ &\le c \max\left\{ m_0^{|\alpha_0|} m_1^{|\alpha_1|} : \alpha_0 \in \mathbb{N}_0^{d_0}, \alpha_1 \in \mathbb{N}_0^d, \frac{|\alpha_0|}{r_0} + \frac{|\alpha_1|}{r} \le 1 \right\} \\ &= c \max\left\{ \left(m_0^{r_0}\right)^{\frac{|\alpha_0|}{r_0}} \left(m_1^{r}\right)^{\frac{|\alpha_1|}{r}} : \alpha_0 \in \mathbb{N}_0^{d_0}, \alpha_1 \in \mathbb{N}_0^d, \frac{|\alpha_0|}{r_0} + \frac{|\alpha_1|}{r} \le 1 \right\} \\ &\le c \max\left(m_0^{r_0}, m_1^r\right) \end{aligned}$$

and therefore

$$c \min(m_1^{-r}, m_0^{-r_0})\, \Psi^0_{m_0,m_1} \subseteq B_{C^{[r_0,r]}(Q_0,Q)}.$$

Arguing as in the proof of the lower bounds of Theorem 4.1 gives the desired result. □

Note that for $r_0 = r$ we recover the results of [2], with the rates

$$\upsilon_5 = \frac{r}{d+d_0}, \quad \upsilon_6 = \begin{cases} \frac{r+\frac{d}{2}}{d+d_0} & \text{if} \quad \frac{r}{d_0} > \frac{1}{2} \\ \frac{r}{d_0} & \text{if} \quad \frac{r}{d_0} \le \frac{1}{2}. \end{cases}$$

Now we compare the exponents υ_5 of the deterministic setting (77) and υ_6 of the randomized setting (78). We assume $r_0 > 0$, otherwise both exponents are zero.

First consider the case $r_0/d_0 > 1/2$. If $r = 0$, then $\upsilon_5 = 0$, $\upsilon_6 = 1/2$, so the randomized rate is by the exponent $1/2$ superior to the (trivial) deterministic one. For $r > 0$ the gap is smaller than $1/2$, but it is never zero. The advantage of randomization can be arbitrarily close to $1/2$ (for large parameter smoothness r_0/d_0 or small t-smoothness r/d).

If $r_0/d_0 \le 1/2$, we have

$$\upsilon_5 = \frac{\frac{r}{d}}{\frac{r_0}{d_0} + \frac{r}{d}} \, \frac{r_0}{d_0}, \quad \upsilon_6 = \frac{r_0}{d_0},$$

so also in this case the gain by randomization is never zero, furthermore, for small r/d it comes close to r_0/d_0, and it reaches this value only for $r = 0$.

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