Complexity of parametric initial value problems in Banach spaces

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Abstract

We consider initial value problems for parameter dependent ordinary differential equations with values in a Banach space and study their complexity both in the deterministic and randomized setting, for input data from various smoothness classes. We develop multilevel algorithms, investigate the convergence of their deterministic and stochastic versions, and prove lower bounds.

1 Introduction and preliminaries

The complexity of initial value problems for ordinary differential equations (ODEs) was studied in [19, 20, 21, 16, 4] for scalar systems and in [15] for the Banach space valued case. In this paper we consider initial value problems for parameter dependent ODEs with values in a Banach space. We study the complexity in the deterministic and randomized setting for various classes of smoothness of the input functions. These classes are closely related to those considered in [6] and include cases of isotropic and of dominating mixed smoothness.

We develop a randomized multilevel algorithm and establish its convergence rate. The deterministic version of it, which is obtained from the randomized one by fixing the random parameters in an arbitrary way, is also studied. The algorithmic approach is a nonlinear analogue of the approximation in [5], based on the multilevel methods of [11, 17]. Furthermore, our analysis uses the Banach space valued generalizations [15] of the scalar results in [16, 4].

We prove lower bounds on the complexity. The algorithm turns out to be of optimal order (up to logarithmic factors) in the deterministic setting. In the randomized setting, for general Banach spaces, there remains an arbitrarily small gap in the exponent. For special spaces like the L_p spaces the convergence rate of the algorithm and the lower bounds are matching also in the randomized setting (again up to some logarithmic factors). This way we obtain almost sharp estimates of the complexity. We also compare the optimal rates of the deterministic and randomized setting, this way assessing the speedup randomization can bring over deterministic methods.

Studying equations in Banach spaces means including finite and infinite systems of scalar ODEs and gives the possibility of considering various norms which are non-equivalent for the case of infinite systems. The Banach space approach is also of interest from the point of view of tractability of high-dimensional problems [28], since the Banach space results imply convergence estimates for finite scalar systems with constants independent of the dimension, see also the comments in Section 6.

Regularity and approximation properties of the solution of parameter dependent initial value problems for ODEs have recently been considered in [10], however, with linear dependence on the parameters and an infinite dimensional parameter space. Complexity of parameter dependent problems was previously studied only for parametric definite integration [17, 12, 31, 5] and for parametric indefinite integration [5]. Both problems are linear, so that in the present paper for the first time the complexity of a nonlinear parametric problem is analyzed.

The paper is organized as follows. In Section 2 we consider Banach space valued ODEs and develop a multilevel approach. The parametric problem is formulated in Section 3 and we show how it fits the Banach space scheme for a single equation of Section 2. In Section 4 the algorithm for the parametric problem is described and convergence rates are derived. Section 5 contains lower bounds and the complexity is established. Finally, in Section 6 we discuss the considered classes and related ones, study special cases of the obtained results, and provide comparisons between deterministic and randomized setting.

Background on Banach space valued differential calculus and ODEs can be found in [1]. For further reading on ODEs in Banach spaces we refer to the monographs [3, 25, 32, 22, 8]. Basic references on information-based complexity theory are [29, 27] and, in particular for the topic of tractability, [28].

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We introduce some notation and concepts from Banach space theory needed in the sequel. For a Banach space X the closed unit ball is denoted by B_X, the open unit ball by B_X^0, the identity mapping on X by I_X, and the dual space by X^*. Given $k \in \mathbb{N}$, Banach spaces X_i $(i = 1, \dots, k)$ and Y, we let $\mathscr{L}(X_1, \dots, X_k, Y)$ be the space of bounded multilinear mappings $T : X_1 \times \dots \times X_k \to Y$ endowed with the canonical norm

$$\|T\|_{\mathscr{L}(X_1,\dots X_k,Y)} = \sup_{x_1 \in B_{X_1},\dots,x_k \in B_{X_k}} \|T(x_1, \dots, x_k)\|.$$

If $X_1 = \dots = X_k = X$, we write $\mathscr{L}_k(X, Y)$. Similarly, if $k = k_1 + k_2$ with $k_1, k_2 \geq 0$, $X_1 = \dots = X_{k_1} = X$, $X_{k_1+1} = \dots = X_{k_1+k_2} = Z$, we use the notation $\mathscr{L}_{k_1,k_2}(X, Z, Y)$. For convenience we extend the notation to $k = 0$ by setting $\mathscr{L}_0(X, Y) = \mathscr{L}_{0,0}(X, Z, Y) = Y$. If $k = 1$, $\mathscr{L}_1(X, Y)$ is the space of bounded linear operators, for which we write $\mathscr{L}(X, Y)$. If $Y = X$, we write $\mathscr{L}(X)$ instead of $\mathscr{L}(X, X)$.

If M is a nonempty set, we let $B(M, X)$ be the space of all X-valued, bounded on M functions, equipped with the supremum norm

$$\|g\|_{B(M,X)} = \sup_{t \in M} \|g(t)\|.$$

If $X = \mathbb{R}$, we write $B(M)$.

Given $1 \le p \le 2$, a Banach space X is said to be of (Rademacher) type p, if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$

$$\mathbb{E} \Big\| \sum_{i=1}^{n} \varepsilon_i x_i \Big\|^p \le c^p \sum_{k=1}^{n} \|x_i\|^p, \tag{1}$$

where $(\varepsilon_i)_{i=1}^n$ is a sequence of independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_i = -1\} = \mathbb{P}\{\varepsilon_i = +1\} = 1/2$ (we refer to [26, 23] for this notion and related facts). The type p constant $\tau_p(X)$ of X is the smallest constant $c > 0$ satisfying (1). If there is no such $c > 0$, we set $\tau_p(X) = \infty$. The space $L_{p_1}(M, \mu)$ with (M, μ) an arbitrary measure space and $p_1 < \infty$ is of type p with $p = \min(p_1, 2)$. Furthermore, there is a constant $c > 0$ such that $\tau_2(\ell_\infty^n) \le c(\log(n+1))^{1/2}$ for all $n \in \mathbb{N}$ (see also Lemma 4.6 below).

Throughout the paper the same symbol $c, c_1, c_2, \dots$ may denote different constants, even in a sequence of relations. The function log always means $\log_2$. For nonnegative reals $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we write $a_n \preceq b_n$ if there are constants $c > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $a_n \le c b_n$. Furthermore, $a_n \asymp b_n$ means that $a_n \preceq b_n$ and $b_n \preceq a_n$. Finally, $a_n \preceq_{\log} b_n$ iff there are constants $c > 0$, $n_0 \in \mathbb{N}$, and $\theta \in \mathbb{R}$ such that for all $n \ge n_0$, $a_n \le c b_n (\log(n+1))^\theta$ and $a_n \asymp_{\log} b_n$ iff $a_n \preceq_{\log} b_n$ and $b_n \preceq_{\log} a_n$.

2 Banach space valued ODEs

Let X and Y be Banach spaces over the reals. This assumption is made because below we consider only real differentiation. Complex spaces can be included by simply considering them as spaces over the reals. Let $-\infty < a < b < +\infty$, $r \in \mathbb{N}_0$, $0 \le \varrho \le 1$, and let $\kappa, L : (0, +\infty) \to (0, +\infty)$ be any functions. We define the following class

$$\mathcal{C}^{r,\varrho}([a,b] \times X, Y; \kappa) \quad \text{of continuous functions} \quad f : [a,b] \times X \to Y$$

having for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ with $|\alpha| = \alpha_1 + \alpha_2 \le r$ continuous partial (Fréchet-) derivatives

$$\frac{\partial^{|\alpha|} f(t,x)}{\partial t^{\alpha_1} \partial x^{\alpha_2}} \in \mathscr{L}_{\alpha_2}(X, Y),$$

such that for all $R > 0$, $t, t_1, t_2 \in [a, b]$, $x, y \in RB_X$, $|\alpha| \le r$

$$\left\| \frac{\partial^{|\alpha|} f(t, x)}{\partial t^{\alpha_1} \partial x^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_2}(X,Y)} \le \kappa(R), \tag{2}$$

and, for $|\alpha| = r$,

$$\left\| \frac{\partial^{|\alpha|} f(t_1, x)}{\partial t^{\alpha_1} \partial x^{\alpha_2}} - \frac{\partial^{|\alpha|} f(t_2, y)}{\partial t^{\alpha_1} \partial x^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_2}(X,Y)} \le \kappa(R)(|t_1 - t_2|^{\varrho} + \|x - y\|^{\varrho}). \tag{3}$$

Moreover, let $\mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a, b] \times X, Y; \kappa, L)$ be the class of all $f \in \mathcal{C}^{r,\varrho}([a, b] \times X, Y; \kappa)$ such that for $R > 0$, $t \in [a, b]$, $x, y \in RB_X$

$$\|f(t, x) - f(t, y)\| \le L(R)\|x - y\|. \tag{4}$$

So the classes introduced above have smoothness (and the Lipschitz property) bounded on bounded sets. If X is finite dimensional, this means local smoothness and local Lipschitz property.

We consider initial value problems for ODEs with values in X

$$u'(t) = f(t, u(t)) \quad (t \in [a, b]), \qquad u(a) = u_0, \tag{5}$$

with $f \in \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a, b] \times X, X; \kappa, L)$ and $u_0 \in X$. A function $u : [a, b] \to X$ is called a solution, if u is continuously differentiable and (5) is satisfied.

Next we introduce the algorithm developed and studied in [15] (and previously, for the scalar case, in [4]). Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, let $t_k = a + kh$ $(k = 0, 1, \ldots, n)$ be the uniform grid on $[a, b]$ of meshsize $h = (b-a)/n$. Furthermore, for $0 \le k \le n-1$ and $1 \le j \le m$ let $P_{k,j}$ be the operator of Lagrange interpolation of degree j on the equidistant grid $t_{k,j,i} = t_k + ih/j$ $(i = 0, \ldots, j)$ on $[t_k, t_{k+1}]$. Let $\xi_1, \ldots, \xi_n$ be independent random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$ such that ξ_k is uniformly distributed on $[t_{k-1}, t_k]$ $(k = 1, \ldots, n)$. Since we will also consider $\xi_k(\omega)$ for fixed $\omega \in \Omega$, we assume (without loss of generality) that

$$\{(\xi_1(\omega), \ldots, \xi_n(\omega)) : \omega \in \Omega\} = [t_0, t_1] \times \cdots \times [t_{n-1}, t_n]. \tag{6}$$

Fix $f \in \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a, b] \times X, X; \kappa, L)$ and $u_0 \in X$, and define $(u_k)_{k=1}^n \subset X$ and X-valued polynomials $p_{k,j}(t)$ for $k = 0, \ldots, n-1$, $j = 0, \ldots, m$ by induction as follows. Assume that $0 \le k \le n-1$ and that u_k is already defined. Then we define $p_{k,0}$ by

$$p_{k,0}(t) \quad = \quad u_k + f(t_k, u_k)(t - t_k) \quad (t \in [t_k, t_{k+1}]). \tag{7}$$

Now suppose $m \ge 1$, $0 \le j < m$, and $p_{k,j}$ is already defined. We define $p_{k,j+1}$ by

$$p_{k,j+1}(t) \quad = \quad u_k + \int_{t_k}^{t} (P_{k,j+1} q_{k,j})(\tau) d\tau, \tag{8}$$

where

$$q_{k,j} \quad = \quad \left(f(t_{k,j+1,i}, p_{k,j}(t_{k,j+1,i}))\right)_{i=0}^{j+1}. \tag{9}$$

Finally, we put

$$u_{k+1} = p_{k,m}(t_{k+1}) + h\left(f(\xi_{k+1}, p_{k,m}(\xi_{k+1})) - p'_{k,m}(\xi_{k+1})\right). \tag{10}$$

The result of the algorithm, the approximation $v \in B([a,b],X)$ to the solution u of (5), is now defined by

$$v(t) = \begin{cases} p_{k,m}(t) & \text{if} \quad t \in [t_k, t_{k+1}) \quad \text{and} \quad 0 \le k \le n-1, \\ u_n & \text{if} \quad t = t_n. \end{cases} \tag{11}$$

Let

$$A_{n,\omega}^m : \mathcal{C}_{\mathrm{Lip}}^{r,\varrho}([a,b] \times X, X; \kappa, L) \times X \to B([a,b],X)$$

denote the resulting mapping, for $\omega \in \Omega$ fixed, that is,

$$A_{n,\omega}^m(f, u_0) = v, \tag{12}$$

and let A_n^m denote the family of mappings $A_n^m = (A_{n,\omega}^m)_{\omega\in\Omega}$. We write $A_n^m(f,u_0)$ for the random variable $(A_{n,\omega}^m(f,u_0))_{\omega\in\Omega}$. Observe that for $m = 0$ we have

$$p_{k,0}(t) \quad = \quad u_k + f(t_k, u_k)(t - t_k) \qquad (t \in [t_k, t_{k+1}],\ 0 \le k \le n-1), \tag{13}$$

$$u_{k+1} \quad = \quad u_k + hf(\xi_{k+1}, p_{k,0}(\xi_{k+1})) \qquad (0 \le k \le n-1). \tag{14}$$

Concerning the definition of $A_{n,\omega}^m$, we note that due to condition (6), fixing any $\omega \in \Omega$ is the same as fixing any values of $\xi_k \in [t_{k-1}, t_k]$ $(k = 1, \dots, n)$. This way we obtain a deterministic algorithm, the ξ_k being fixed algorithm parameters.

Given also $\sigma, \lambda > 0$, we let $\mathcal{F}^{r,\varrho}([a,b] \times X, X; \kappa, L, \sigma, \lambda)$ be the class of all pairs (f, u_0) with $f \in \mathcal{C}_{\mathrm{Lip}}^{r,\varrho}([a,b] \times X, X; \kappa, L)$, $u_0 \in \sigma B_X$, such that the initial value problem (5) has a solution u (which is unique, due to assumption (4)) satisfying

$$\|u\|_{B([a,b],X)} \le \lambda. \tag{15}$$

If $r = \varrho = 0$, we require, in addition, that (f, u_0) is such that for all $n \in \mathbb{N}$, $\omega \in \Omega$

$$\|A_{n,\omega}^0(f, u_0)\|_{B([a,b],X)} \le \lambda. \tag{16}$$

The solution operator

$$\mathcal{S} : \mathcal{F}^{r,\varrho}([a,b] \times X, X; \kappa, L, \sigma, \lambda) \to B([a,b],X)$$

is defined for $(f, u_0) \in \mathcal{F}^{r,\varrho}([a,b] \times X, X; \kappa, L, \sigma, \lambda)$ by $\mathcal{S}(f, u_0) = u$, where u is the solution of the initial value problem (5).

Proposition 2.1. *Let $r \in \mathbb{N}_0$, $0 \le \varrho \le 1$, $\kappa, L : (0,+\infty) \to (0,+\infty)$, $\sigma, \lambda > 0$, $1 \le p \le 2$, and let $m \in \mathbb{N}_0$ if $r+\varrho > 0$ and $m = 0$ if $r = \varrho = 0$. Then there are constants $c_1, c_2 > 0$ and $\nu_0 \in \mathbb{N}$ such that for all Banach spaces X and all $n \ge \nu_0$*

$$\sup_{(f,u_0)\in\mathcal{F}^{r,\varrho}([a,b]\times X,X;\kappa,L,\sigma,\lambda)} \|\mathcal{S}(f,u_0) - A^m_{n,\omega}(f,u_0)\|_{B([a,b],X)} \le c_1 n^{-\min(r+\varrho,m+1)} \qquad (\omega\in\Omega) \tag{17}$$

and

$$\sup_{(f,u_0)\in\mathcal{F}^{r,\varrho}([a,b]\times X,X;\kappa,L,\sigma,\lambda)} \Big(\mathbb{E}\,\|\mathcal{S}(f,u_0) - A^m_{n,\omega}(f,u_0)\|^p_{B([a,b],X)}\Big)^{1/p} \le c_2\tau_p(X) n^{-\min(r+\varrho,m+1)-1+1/p}. \tag{18}$$

Proof. We put

$$U = [a,b]\times(\lambda+1)B^0_X, \quad U_0 = \sigma B_X, \quad V = [a,b]\times\lambda B_X.$$

Let $(f,u_0) \in \mathcal{F}^{r,\varrho}([a,b]\times X, X;\kappa,L,\sigma,\lambda)$. First we consider the case $r+\varrho>0$. By (15) we have, in the notation of [15],

$$(f|_U, u_0) \in \mathcal{F}^{r,\varrho}(U,\kappa(\lambda+1),L(\lambda+1),U_0,V).$$

Since $\lambda B_X + \frac{1}{2}B_X \subset (\lambda+1)B^0_X$, Theorem 3.3 of [15] gives (17–18). Now let $r=\varrho=0$ and put $u = S(f,u_0)$. Then for $t\in[t_k,t_{k+1}]$

$$u(t_k) + \kappa(\lambda+1)(t-t_k)B_X \subseteq \lambda B_X + \kappa(\lambda+1)\frac{b-a}{n}B_X \subseteq (\lambda+1)B^0_X$$

whenever $n \ge \nu_0 := \lfloor \kappa(\lambda+1)(b-a)\rfloor + 1$. Taking into account (15–16), we see that, in the notation of [15],

$$(f|_U,u_0) \in \mathscr{H}^{0,0}(U,\kappa(\lambda+1),L(\lambda+1),U_0,V,0,n) \quad (n\ge\nu_0).$$

Therefore (17–18) follow for $n \ge \nu_0$ from Proposition 3.4 of [15]. □

In the sequel we need the following result.

Lemma 2.2. *Let Z and Z_1 be Banach spaces, $f \in \mathcal{C}^{0,0}_{\mathrm{Lip}}([a,b]\times Z, Z;\kappa,L)$, and $T \in \mathscr{L}(Z,Z_1)$. Assume that there are $\kappa_1, L_1 : (0,+\infty)\to(0,+\infty)$ and a function $g \in \mathcal{C}^{0,0}_{\mathrm{Lip}}([a,b]\times Z_1, Z_1;\kappa_1,L_1)$ such that for all $t\in[a,b]$, $z \in Z$*

$$Tf(t,z) = g(t,Tz). \tag{19}$$

Then for all $u_0\in Z$ the following hold. For $m\in\mathbb{N}_0$, $n\in\mathbb{N}$, $\omega\in\Omega$

$$TA^m_{n,\omega}(f,u_0) = A^m_{n,\omega}(g,Tu_0). \tag{20}$$

Moreover, if u is a solution of (5), then Tu is a solution of the ODE in Z_1

$$w'(t) = g(t,w(t)) \quad (t\in[a,b]), \qquad w(a) = Tu_0. \tag{21}$$

Proof. Applying T to (5), we get

$$\begin{aligned} (Tu(t))' &= Tu'(t) = Tf(t,u(t)) = g(t,Tu(t)) \quad (t \in [a,b]) \\ Tu(a) &= Tu_0. \end{aligned}$$

Now the second statement follows from uniqueness of the solution of (21).

Let u_k, $p_{k,j}$, and $q_{k,j}$ be the resulting sequences (7–10), when applying $A^m_{n,\omega}$ to (f, u_0). Furthermore, put $\tilde{u}_0 = Tu_0$ and let $\tilde{u}_k$, $\tilde{p}_{k,j}$, and $\tilde{q}_{k,j}$ be the respective functions from applying $A^m_{n,\omega}$ to $(g, \tilde{u}_0)$. We show that for $0 \le k \le n$

$$Tu_k = \tilde{u}_k \tag{22}$$

and for $0 \le k \le n-1$

$$Tp_{k,j} = \tilde{p}_{k,j} \quad (0 \le j \le m). \tag{23}$$

First we prove that given k with $0 \le k \le n-1$, (22) implies (23). So assume that (22) holds. We show (23) by induction over j. Let $j = 0$. By (19) and (22),

$$Tf(t_k, u_k) = g(t_k, Tu_k) = g(t_k, \tilde{u}_k),$$

therefore

$$Tp_{k,0}(t) = Tu_k + Tf(t_k,u_k)(t - t_k) = \tilde{u}_k + g(t_k, \tilde{u}_k)(t - t_k) = \tilde{p}_{k,0}(t).$$

Now we assume that (23) holds for some j with $0 \le j < m$. Then

$$Tp_{k,j}(t_{k,j+1,i}) = \tilde{p}_{k,j}(t_{k,j+1,i}) \quad (i = 0, \ldots, j+1).$$

It follows that

$$Tf(t_{k,j+1,i}, p_{k,j}(t_{k,j+1,i})) = g(t_{k,j+1,i}, \tilde{p}_{k,j}(t_{k,j+1,i}))$$

and consequently

$$\begin{aligned} Tp_{k,j+1}(t) &= Tu_k + T\int_{t_k}^t (P_{k,j+1}q_{k,j+1})(\tau)d\tau \\ &= \tilde{u}_k + \int_{t_k}^t (P_{k,j+1}\tilde{q}_{k,j+1})(\tau)d\tau = \tilde{p}_{k,j+1}(t). \end{aligned}$$

This completes the induction over j and the proof that (22) implies (23).

Next we show (22) by induction over k. For $k = 0$ it holds by definition. Now suppose (22) and thus (23) hold for some k with $0 \le k \le n-1$. It follows that

$$\begin{aligned} Tu_{k+1} &= Tp_{k,m}(t_{k+1}) + h\big(Tf(\xi_{k+1}, p_{k,m}(\xi_{k+1})) - Tp'_{k,m}(\xi_{k+1})\big) \\ &= \tilde{p}_{k,m}(t_{k+1}) + h\big(g(\xi_{k+1}, \tilde{p}_{k,m}(\xi_{k+1})) - \tilde{p}'_{k,m}(\xi_{k+1})\big) = \tilde{u}_{k+1}. \end{aligned}$$

This shows (22) for $k+1$, completes the induction over k, and proves (22–23). Now (20) follows from (22–23) and (11–12).

□

Now we develop a multilevel procedure. Assume that a Banach space Y is continuously embedded into the Banach space X, and let J be the embedding map. We shall identify elements of Y with their images in X. Let $r, r_1 \in \mathbb{N}_0$, $0 \le \varrho, \varrho_1 \le 1$, $\kappa, L : (0,+\infty) \to (0,+\infty)$, $\sigma, \lambda > 0$, and consider the set

$$\mathcal{K} = \mathcal{F}^{r,\varrho}([a,b] \times X, X; \kappa, L, \sigma, \lambda) \cap \mathcal{F}^{r_1,\varrho_1}([a,b] \times Y, Y; \kappa, L, \sigma, \lambda), \tag{24}$$

which is the set of all $(f, u_0) \in \mathcal{F}^{r,\varrho}([a,b] \times X, X; \kappa, L, \sigma, \lambda)$ such that f maps $[a,b] \times Y$ to Y and, if f is considered as such a mapping, (f, u_0) belongs to $\mathcal{F}^{r_1,\varrho_1}([a,b] \times Y, Y; \kappa, L, \sigma, \lambda)$.

Observe that the solution operator $\mathcal{S}$ is correctly defined also on $\mathcal{K}$, since the respective operators on $\mathcal{F}^{r,\varrho}([a,b] \times X, X; \kappa, L, \sigma, \lambda)$ and $\mathcal{F}^{r_1,\varrho_1}([a,b] \times Y, Y; \kappa, L, \sigma, \lambda)$ coincide on the intersection. This follows from Lemma 2.2 with $Z = Y$, $Z_1 = X$, $T = J$, and $g = f$.

Let $(P_l)_{l=0}^{\infty} \subset \mathscr{L}(X)$ and fix any $l_0, l_1 \in \mathbb{N}_0$, $l_0 \le l_1$, and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$. For $(f, u_0) \in \mathcal{K}$ and $\omega \in \Omega$ we define an approximation $A_\omega(f, u_0)$ to $u = \mathcal{S}(f, u_0)$ in the space $B([a,b], X)$ as follows

$$A_\omega(f, u_0) \;=\; P_{l_0} A^r_{n_{l_0},\omega}(f, u_0) + \sum_{l=l_0+1}^{l_1} (P_l - P_{l-1}) A^{r_1}_{n_l,\omega}(f, u_0). \tag{25}$$

(Here we assume that the underlying probability space $(\Omega, \Sigma, \mathbb{P})$ is such that all random variables required on the levels $l_0, \dots, l_1$ are defined on it.)

We assume that there is a constant $\gamma_0 > 0$ such that for all $l \in \mathbb{N}_0$

$$\|P_l\|_{\mathscr{L}(X)} \;\le\; \gamma_0. \tag{26}$$

Furthermore, we assume the existence of a family of operators $(T_l)_{l=0}^{\infty} \subset \mathscr{L}(X)$ with the following properties. There are constants $\gamma_1, \gamma_2 > 0$ such that for $l \in \mathbb{N}_0$

$$\|T_l\|_{\mathscr{L}(X)} \le \gamma_1, \tag{27}$$

T_l maps Y to Y,

$$\|T_l\|_{\mathscr{L}(Y)} \le \gamma_2, \tag{28}$$

and

$$P_k T_l = P_k \quad (k \le l). \tag{29}$$

Finally, let $\mathcal{K}_0 \subseteq \mathcal{K}$ be a subset with the following property: If f is such that there exists a u_0 with $(f, u_0) \in \mathcal{K}_0$, then

$$T_l f(t, x) = T_l f(t, T_l x) \quad (t \in [a,b], x \in X, l \in \mathbb{N}_0). \tag{30}$$

We put

$$X_l = \mathrm{cl}_X(T_l(X)), \quad Y_l = \mathrm{cl}_Y(T_l(Y)) \quad (l \in \mathbb{N}_0),$$

where cl denotes the closure in the respective space.

Note that the T_l do not enter the algorithm definition, they are needed for the error analysis. Furthermore, (27–30) hold, in particular, for $\mathcal{K}_0 = \mathcal{K}$ and $T_l \equiv I_X$. In this case the error estimate (32) in the randomized setting of Proposition 2.3 below requires some type assumption on the spaces X and Y. However, in Sections 3 and 4 we shall consider spaces X and Y which have no nontrivial type, while certain finite dimensional subspaces related to the approximation do have type constants with nontrivial estimates. Therefore we will also consider other choices of $\mathcal{K}_0$ and T_l, see Section 4.

Proposition 2.3. *Let $r, r_1 \in \mathbb{N}_0$, $0 \le \varrho, \varrho_1 \le 1$, $\kappa, L : (0, +\infty) \to (0, +\infty)$, $\sigma, \lambda, \gamma_{0-2} > 0$, and $1 \le p \le 2$. Then there are constants $c_1, c_2 > 0$ and $\nu_0 \in \mathbb{N}$ such that the following holds.*

Given Banach spaces X, Y with Y continuously embedded into X, sequences $(P_l)_{l=0}^{\infty}, (T_l)_{l=0}^{\infty} \subset \mathscr{L}(X)$ satisfying (26–29), let $\mathcal{K}$ be defined by (24), and let $\mathcal{K}_0 \subseteq \mathcal{K}$ be such that (30) is fulfilled. Then for all $l_0, l_1 \in \mathbb{N}_0$ with $l_0 \le l_1$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ with $n_l \ge \nu_0$ $(l_0 < l \le l_1)$ the so-defined algorithm (A_ω) satisfies

$$\begin{aligned}
&\sup_{(f,u_0)\in\mathcal{K}_0} \|\mathcal{S}(f,u_0) - A_\omega(f,u_0)\|_{B([a,b],X)} \\
&\le \; c_1\|J - P_{l_1}J\|_{\mathscr{L}(Y,X)} + c_1 n_{l_0}^{-r-\varrho} \\
&\quad + c_1 \sum_{l=l_0+1}^{l_1} \|(P_l - P_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-r_1-\varrho_1} \qquad (\omega \in \Omega) \qquad (31)
\end{aligned}$$

and, for any $l^* \in \mathbb{N}_0$ with $l_0 \le l^* \le l_1$

$$\begin{aligned}
&\sup_{(f,u_0)\in\mathcal{K}_0} \left(\mathbb{E}\,\|\mathcal{S}(f,u_0) - A_\omega(f,u_0)\|_{B([a,b],X)}^p\right)^{1/p} \\
&\le \; c_2\|J - P_{l_1}J\|_{\mathscr{L}(Y,X)} + c_2\tau_p(X_{l_0}) n_{l_0}^{-r-\varrho-1+1/p} \\
&\quad + c_2 \sum_{l=l_0+1}^{l^*} \tau_p(Y_l)\|(P_l - P_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-r_1-\varrho_1-1+1/p} \\
&\quad + c_2 \sum_{l=l^*+1}^{l_1} \|(P_l - P_{l-1})J\|_{\mathscr{L}(Y,X)} n_l^{-r_1-\varrho_1}. \qquad (32)
\end{aligned}$$

Remark 2.4. Note that the natural case of estimate (32) would be $l^* = l_1$, and it is this case which we use in this paper. However, as in [6], the more general approach will be used in [7] to determine sharp rates, including precise powers of logarithms.

Proof. Let $(f, u_0) \in \mathcal{K}_0$. Then by (24) and (15)

$$\|\mathcal{S}(f,u_0)\|_{B([a,b],Y)} \le \lambda.$$

It follows that

$$\|\mathcal{S}(f,u_0)-P_{l_1}\mathcal{S}(f,u_0)\|_{B([a,b],X)} \;\le\; \lambda\|J-P_{l_1}J\|_{\mathscr{L}(Y,X)}. \tag{33}$$

We have by (27) and (28)

$$T_{l_0}f \;\in\; \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a,b]\times X_{l_0},X_{l_0};\gamma_1\kappa,\gamma_1 L) \tag{34}$$

$$T_l f \;\in\; \mathcal{C}^{r_1,\varrho_1}_{\mathrm{Lip}}([a,b]\times Y_l,Y_l;\gamma_2\kappa,\gamma_2 L) \quad (l_0<l\le l_1), \tag{35}$$

and therefore, using (30) and Lemma 2.2 with $g=T_l f$,

$$T_l\mathcal{S}(f,u_0) \;=\; \mathcal{S}(T_l f,T_l u_0) \quad (l_0\le l\le l_1) \tag{36}$$

$$T_{l_0}A^r_{n_{l_0},\omega}(f,u_0) \;=\; A^r_{n_{l_0},\omega}(T_{l_0}f,T_{l_0}u_0) \quad (\omega\in\Omega) \tag{37}$$

$$T_l A^{r_1}_{n_l,\omega}(f,u_0) \;=\; A^{r_1}_{n_l,\omega}(T_l f,T_l u_0) \quad (\omega\in\Omega,\ l_0<l\le l_1). \tag{38}$$

This together with (27–28) and (34–35) implies

$$(T_{l_0}f,T_{l_0}u_0) \;\in\; \mathcal{F}^{r,\varrho}([a,b]\times X_{l_0},X_{l_0};\gamma_1\kappa,\gamma_1 L,\gamma_1\sigma,\gamma_1\lambda) \tag{39}$$

$$(T_l f,T_l u_0) \;\in\; \mathcal{F}^{r_1,\varrho_1}([a,b]\times Y_l,Y_l;\gamma_2\kappa,\gamma_2 L,\gamma_2\sigma,\gamma_2\lambda) \quad (l_0<l\le l_1). \tag{40}$$

By (25),

$$\begin{aligned}
&\|\mathcal{S}(f,u_0)-A_\omega(f,u_0)\|_{B([a,b],X)}\\
&\le \|\mathcal{S}(f,u_0)-P_{l_1}\mathcal{S}(f,u_0)\|_{B([a,b],X)}\\
&\quad+\|P_{l_0}\mathcal{S}(f,u_0)-P_{l_0}A^r_{n_{l_0},\omega}(f,u_0)\|_{B([a,b],X)}\\
&\quad+\sum_{l=l_0+1}^{l_1}\|(P_l-P_{l-1})(\mathcal{S}(f,u_0)-A^{r_1}_{n_l,\omega}(f,u_0))\|_{B([a,b],X)}.
\end{aligned} \tag{41}$$

Furthermore, by (36), (37), and (26),

$$\begin{aligned}
&\|P_{l_0}\mathcal{S}(f,u_0)-P_{l_0}A^r_{n_{l_0},\omega}(f,u_0)\|_{B([a,b],X)}\\
&= \|P_{l_0}T_{l_0}\mathcal{S}(f,u_0)-P_{l_0}T_{l_0}A^r_{n_{l_0},\omega}(f,u_0)\|_{B([a,b],X)}\\
&= \|P_{l_0}\mathcal{S}(T_{l_0}f,T_{l_0}u_0)-P_{l_0}A^r_{n_{l_0},\omega}(T_{l_0}f,T_{l_0}u_0)\|_{B([a,b],X)}\\
&\le \gamma_0\|\mathcal{S}(T_{l_0}f,T_{l_0}u_0)-A^r_{n_{l_0},\omega}(T_{l_0}f,T_{l_0}u_0)\|_{B([a,b],X_{l_0})}
\end{aligned} \tag{42}$$

and similarly, by (36) and (38)

$$\begin{aligned}
&\|(P_l-P_{l-1})(\mathcal{S}(f,u_0)-A^{r_1}_{n_l,\omega}(f,u_0))\|_{B([a,b],X)}\\
&= \|(P_l-P_{l-1})T_l(\mathcal{S}(f,u_0)-A^{r_1}_{n_l,\omega}(f,u_0))\|_{B([a,b],X)}\\
&= \|(P_l-P_{l-1})(\mathcal{S}(T_l f,T_l u_0)-A^{r_1}_{n_l,\omega}(T_l f,T_l u_0))\|_{B([a,b],X)}\\
&\le \|(P_l-P_{l-1})J\|_{\mathscr{L}(Y,X)}\|\mathcal{S}(T_l f,T_l u_0)-A^{r_1}_{n_l,\omega}(T_l f,T_l u_0)\|_{B([a,b],Y_l)}.
\end{aligned} \tag{43}$$

By (39), (40), and Proposition 2.1, for all $\omega \in \Omega$ and $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ with $n_l \geq \nu_0$ $(l_0 \leq l \leq l_1)$

$$\|\mathcal{S}(T_{l_0}f, T_{l_0}u_0) - A^r_{n_{l_0},\omega}(T_{l_0}f, T_{l_0}u_0)\|_{B([a,b],X_{l_0})} \leq c n_{l_0}^{-r-\varrho} \tag{44}$$

$$\|\mathcal{S}(T_l f, T_l u_0) - A^{r_1}_{n_l,\omega}(T_l f, T_l u_0)\|_{B([a,b],Y_l)} \leq c n_l^{-r_1-\varrho_1} \tag{45}$$

and

$$\Big(\mathbb{E}\,\|\mathcal{S}(T_{l_0}f, T_{l_0}u_0) - A^r_{n_{l_0},\omega}(T_{l_0}f, T_{l_0}u_0)\|^p_{B([a,b],X_{l_0})}\Big)^{1/p} \leq c\tau_p(X_{l_0}) n_{l_0}^{-r-\varrho-1+1/p} \tag{46}$$

$$\Big(\mathbb{E}\,\|\mathcal{S}(T_l f, T_l u_0) - A^{r_1}_{n_l,\omega}(T_l f, T_l u_0)\|^p_{B([a,b],Y_l)}\Big)^{1/p} \leq c\tau_p(Y_l) n_l^{-r_1-\varrho_1-1+1/p}. \tag{47}$$

Combining (33) and (41–45) yields (31). Relation (32) follows in a similar way from (33), (41–43), and (45–47).

□

3 The parametric problem as a Banach space valued ODE

Let $d_0 \in \mathbb{N}$, $Q = [0,1]^{d_0}$. To keep notation consistent, instead of considering derivatives with respect to single components of $s \in \mathbb{R}^{d_0}$, we consider derivatives with respect to the vector s, in the sense of calculus on vector spaces as in the previous section. So below $\frac{df}{ds}$ is the Jacobian, $\frac{d^2f}{ds^2}$ the Hessian, etc. The space $\mathbb{R}^{d_0}$ is equipped with the Euclidean norm. For $r \in \mathbb{N}_0$ and Z a Banach space we let $C^r(Q,Z)$ be the space of Z-valued r-times continuously differentiable functions on Q, endowed with the norm

$$\|f\|_{C^r(Q,Z)} = \max_{0\leq j\leq r} \sup_{s\in Q} \left\| \frac{d^j f(s)}{ds^j} \right\|_{\mathscr{L}_j(\mathbb{R}^{d_0},Z)}.$$

Note that for $r \geq 1$ this is not the standard norm on $C^r(Q,Z)$ (given by the maximum of the supremum-norms of the partial derivatives with respect to the components of s), but it is equivalent, with a constant depending only on d_0 and r. We use the notation $C^r(Q)$ if $Z = \mathbb{R}$. Furthermore, $C^0(Q,Z)$ is understood to be the space of continuous functions on Q, for which we write $C(Q,Z)$ and $C(Q)$ if $Z = \mathbb{R}$.

Given functions $\kappa, L : (0,+\infty) \to (0,+\infty)$, $r_0, r \in \mathbb{N}_0$, $0 \leq \varrho \leq 1$, and Banach spaces Z, Z_1, we define the following class

$\mathscr{C}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa)$ of continuous functions $f : Q \times [a,b] \times Z \to Z_1$

having for $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}_0^3$ with $\alpha_0 \le r_0$, $\alpha_1 \le r$, and $\alpha_0 + \alpha_1 + \alpha_2 \le r_0 + r$ continuous partial derivatives

$$\frac{\partial^{|\alpha|} f(s,t,z)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} \in \mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0}, Z, Z_1)$$

satisfying for $R > 0$, $s \in Q$, $t \in [a,b]$, $z \in RB_Z$

$$\left\| \frac{\partial^{|\alpha|} f(s,t,z)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0},Z,Z_1)} \le \kappa(R) \tag{48}$$

and for $s \in Q$, $t_1, t_2 \in [a,b]$, $z_1, z_2 \in RB_Z$

$$\begin{aligned} &\left\| \frac{\partial^{|\alpha|} f(s,t_1,z_1)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} - \frac{\partial^{|\alpha|} f(s,t_2,z_2)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0},Z,Z_1)} \\ &\qquad \le \kappa(R)|t_1 - t_2|^{\varrho} + \kappa(R)\|z_1 - z_2\|^{\varrho}. \end{aligned} \tag{49}$$

Moreover, we let $\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa, L)$ be the class of all $f \in \mathscr{C}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa)$ satisfying for $\alpha = (\alpha_0, 0, \alpha_2)$ with $\alpha_0 + \alpha_2 \le r_0$, $R > 0$, $s \in Q$, $t \in [a,b]$, $z_1, z_2 \in RB_Z$

$$\left\| \frac{\partial^{|\alpha|} f(s,t,z_1)}{\partial s^{\alpha_0} \partial z^{\alpha_2}} - \frac{\partial^{|\alpha|} f(s,t,z_2)}{\partial s^{\alpha_0} \partial z^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0},Z,Z_1)} \le L(R)\|z_1 - z_2\|. \tag{50}$$

Clearly, if $r_0', r' \in \mathbb{N}_0$ are such that $r_0' \le r_0$, $r' \le r$, then

$$\mathscr{C}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa) \subseteq \mathscr{C}^{r_0',r',\varrho}(Q \times [a,b] \times Z, Z_1; \kappa) \tag{51}$$

$$\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa, L) \subseteq \mathscr{C}_{\mathrm{Lip}}^{r_0',r',\varrho}(Q \times [a,b] \times Z, Z_1; \kappa, L). \tag{52}$$

Furthermore, if $\varrho' \le \varrho$, then

$$\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa, L) \subseteq \mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho'}(Q \times [a,b] \times Z, Z_1; 2\kappa, L), \tag{53}$$

where the factor 2 comes from the case $\max(|t_1 - t_2|, \|z_1 - z_2\|) > 1$, in which (48) with constant κ trivially implies (49) with constant 2κ. Integration yields

$$\mathscr{C}_{\mathrm{Lip}}^{r_0,r+1,0}(Q \times [a,b] \times Z, Z_1; \kappa, L) \subseteq \mathscr{C}_{\mathrm{Lip}}^{r_0,r,1}(Q \times [a,b] \times Z, Z_1; \kappa, L). \tag{54}$$

Finally note that it would suffice to require (49) and (50) for certain subsets of the sets of multiindices α to obtain (up to constants) the same classes – we omit the details, because the definition given above is more convenient for us.

The classes above were introduced for two Banach spaces Z, Z_1. Some of the lemmas below will be formulated in this general form, for technical convenience. However, for the formulation of the problem and later for the main results we have $Z_1 = Z$.

Now we consider the numerical solution of initial value problems for Z-valued ODEs depending on a parameter $s \in Q$

$$\begin{aligned} \frac{d}{dt}u(s,t) &= f(s,t,u(s,t)) \quad (s \in Q, t \in [a,b]) \qquad (55)\\ u(s,a) &= u_0(s) \quad (s \in Q) \qquad (56)\end{aligned}$$

with $f \in \mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z; \kappa, L)$ and $u_0 \in C^{r_0}(Q,Z)$. A function $u : Q \times [a,b] \to Z$ is called a solution if for each $s \in Q$, $u(s,t)$ is continuously differentiable as a function of t and (55–56) are satisfied.

The class $\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z; \kappa, L)$ introduced above is a certain class of functions with dominating mixed smoothness. We will consider the intersection of two such classes. This enables us to exploit the full generality of (24) and, in particular, to include also functions with isotropic smoothness. We refer to Section 6 for further motivation, discussion, and special cases of this choice. To define the parametric problem, let $r_1 \in \mathbb{N}_0$, $0 \le \varrho_1 \le 1$, $\sigma, \lambda > 0$, and let $\mathscr{F}$ be the class of all

$$\begin{aligned} (f,u_0) &= \big(\mathscr{C}_{\mathrm{Lip}}^{0,r,\varrho}(Q \times [a,b] \times Z, Z; \kappa, L) \cap \mathscr{C}_{\mathrm{Lip}}^{r_0,r_1,\varrho_1}(Q \times [a,b] \times Z, Z; \kappa, L)\big) \\ &\quad \times \sigma B_{C^{r_0}(Q,Z)} \qquad (57)\end{aligned}$$

such that the parameter dependent initial value problem (55–56) has a solution $u(s,t)$ (which is unique, due to the assumption (57) on f) such that

$$\sup_{s \in Q,\, t \in [a,b]} \|u(s,t)\| \le \lambda, \qquad (58)$$

and moreover, if $r = \varrho = r_1 = \varrho_1 = 0$, then for all $n \in \mathbb{N}$, $\omega \in \Omega$

$$\sup_{s \in Q} \left\| A_{n,\omega}^0(f_s, u_0(s)) \right\|_{B([a,b],Z)} \le \lambda, \qquad (59)$$

where f_s for fixed $s \in Q$ denotes the function $f(s,\cdot,\cdot)$ from $[a,b] \times Z$ to Z. We define the solution operator

$$\mathscr{S} : \mathscr{F} \to B(Q \times [a,b], Z) \qquad (60)$$

for $(f,u_0) \in \mathscr{F}$ by $\mathscr{S}(f,u_0) = u$, where $u = u(s,t)$ is the solution of (55–56).

For a continuous function $g : Q \times [a,b] \times Z \to Z_1$ we define a function

$$\bar{g} : [a,b] \times C(Q,Z) \to C(Q,Z_1)$$

by setting for $t \in [a,b]$, $x \in C(Q,Z)$

$$(\bar{g}(t,x))(s) = g(s,t,x(s)) \quad (s \in Q).$$

The following is the central result of this section. It relates the parametric problem to the problem of a single Banach space valued ODE considered in Section 2, with $X = C(Q,Z)$ and $Y = C^{r_0}(Q,Z)$.

Proposition 3.1. *Given $r_0, r, r_1 \in \mathbb{N}_0$, $0 \le \varrho, \varrho_1 \le 1$, functions $\kappa, L : (0,+\infty) \to (0,+\infty)$, $\sigma, \lambda > 0$, there are $\lambda_1 > 0$ and $\kappa_1, L_1 : (0,+\infty) \to (0,+\infty)$ such that the following holds. Let Z be a Banach space and let $\mathscr{F}$ be defined by* (57)*. Then for all $(f, u_0) \in \mathscr{F}$*

$$\begin{aligned}(\bar f, u_0) \;\in\; & \mathcal{F}^{r,\varrho}([a,b] \times C(Q,Z), C(Q,Z); \kappa_1, L_1, \sigma, \lambda_1) \\ & \cap \mathcal{F}^{r_1,\varrho_1}([a,b] \times C^{r_0}(Q,Z), C^{r_0}(Q,Z); \kappa_1, L_1, \sigma, \lambda_1)\end{aligned}$$

and

$$\mathcal{S}(\bar f, u_0) = \mathscr{S}(f, u_0). \tag{61}$$

Concerning relation (61), we note that we identify functions from $B(Q \times [a,b], Z)$, $u = u(s,t)$, with functions from $B([a,b], B(Q,Z))$, $u(t) = u(\,\cdot\,, t)$. For the proof of Proposition 3.1 we need a number of lemmas. We emphasize that the constants (including the functions κ_1, L_1) in the lemmas of this section do not depend on Z and Z_1.

Lemma 3.2. *Given κ, L, there are functions $\kappa_1, L_1 : (0,+\infty) \to (0,+\infty)$ such that the following holds: for all $f \in \mathscr{C}^{r_0,0,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa)$, $\bar f$ maps $[a,b] \times C^{r_0}(Q,Z)$ to $C^{r_0}(Q,Z_1)$ and, considered as such a mapping, satisfies*

$$\bar f \in \mathcal{C}^{0,\varrho}([a,b] \times C^{r_0}(Q,Z), C^{r_0}(Q,Z_1); \kappa_1) \tag{62}$$

and, if $f \in \mathscr{C}^{r_0,0,\varrho}_{\mathrm{Lip}}(Q \times [a,b] \times Z, Z_1; \kappa, L)$, then

$$\bar f \in \mathcal{C}^{0,\varrho}_{\mathrm{Lip}}([a,b] \times C^{r_0}(Q,Z), C^{r_0}(Q,Z_1); \kappa_1, L_1). \tag{63}$$

Proof. We argue by induction over $r_0 \in \mathbb{N}_0$. Let $r_0 = 0$. First we show that if $g : Q \times [a,b] \times Z \to Z_1$ is a continuous function, then $\bar g$ is continuous from $[a,b] \times C(Q,Z)$ to $C(Q,Z_1)$. Let $t, t_n \in [a,b]$, $x, x_n \in C(Q,Z)$ ($n \in \mathbb{N}$) be such that

$$\lim_{n\to\infty} |t_n - t| = 0, \quad \lim_{n\to\infty} \|x_n - x\|_{C(Q,Z)} = 0.$$

It follows that

$$K = \{x_n(s) : \; s \in Q,\, n \in \mathbb{N}\} \cup \{x(s) : \; s \in Q\}$$

is a compact subset of Z. Consequently, g is uniformly continuous on $Q \times [a,b] \times K$ and therefore

$$\lim_{n\to\infty} \sup_{s\in Q} \|g(s, t_n, x_n(s)) - g(s, t, x(s))\|_{Z_1} = 0,$$

which is the continuity of $\bar g$. The boundedness, Hölder, and Lipschitz properties of $\bar f$ are readily checked on the basis of those for f. This completes the proof of the case $r_0 = 0$.

Now let $r_0 \geq 1$ and assume that the statements (62) and (63) hold for $r_0 - 1$. We start with (62). Let $f \in \mathscr{C}^{r_0,0,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa)$. Then by (48–51),

$$\begin{aligned} f &\in \mathscr{C}^{r_0-1,0,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa) \\ g_1 := \frac{\partial f}{\partial s} &\in \mathscr{C}^{r_0-1,0,\varrho}(Q \times [a,b] \times Z, \mathscr{L}(\mathbb{R}^{d_0}, Z_1); \kappa) \\ g_2 := \frac{\partial f}{\partial z} &\in \mathscr{C}^{r_0-1,0,\varrho}(Q \times [a,b] \times Z, \mathscr{L}(Z, Z_1); \kappa), \end{aligned}$$

therefore, by the induction assumption,

$$\bar{f} \in \mathcal{C}^{0,\varrho}([a,b] \times C^{r_0-1}(Q,Z), C^{r_0-1}(Q,Z_1); \kappa_1) \tag{64}$$

$$\bar{g}_1 \in \mathcal{C}^{0,\varrho}([a,b] \times C^{r_0-1}(Q,Z), C^{r_0-1}(Q, \mathscr{L}(\mathbb{R}^{d_0}, Z_1)); \kappa_1) \tag{65}$$

$$\bar{g}_2 \in \mathcal{C}^{0,\varrho}([a,b] \times C^{r_0-1}(Q,Z), C^{r_0-1}(Q, \mathscr{L}(Z, Z_1)); \kappa_1). \tag{66}$$

Fix $t \in [a,b]$ and $x \in C^{r_0}(Q,Z)$. Then

$$\left(\frac{d}{ds}\bar{f}(t,x)\right)(s) = g_1(s,t,x(s)) + g_2(s,t,x(s))\frac{dx(s)}{ds},$$

which means that

$$\frac{d}{ds}\bar{f}(t,x) = \bar{g}_1(t,x) + \bar{g}_2(t,x)\frac{dx}{ds}. \tag{67}$$

(64–67) readily imply that $\bar{f}$ maps $[a,b] \times C^{r_0}(Q,Z)$ to $C^{r_0}(Q,Z_1)$ and $\bar{f}$ is a continuous function from $[a,b] \times C^{r_0}(Q,Z)$ to $C^{r_0}(Q,Z_1)$. We omit the proof, since it goes along the same lines as the argument below.

Now we show that $\bar{f}$ satisfies the boundedness and the Hölder condition for r_0. Let $R > 0$ and

$$x, y \in RB_{C^{r_0}(Q,Z)}. \tag{68}$$

This implies

$$\left\|\frac{dx}{ds}\right\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))} \leq R \tag{69}$$

and together with (64–66)

$$\left\|\bar{f}(t,x)\right\|_{C^{r_0-1}(Q,Z_1)} \leq \kappa_1(R) \tag{70}$$

$$\|\bar{g}_1(t,x)\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} \leq \kappa_1(R)$$

$$\|\bar{g}_2(t,x)\|_{C^{r_0-1}(Q,\mathscr{L}(Z,Z_1))} \leq \kappa_1(R), \tag{71}$$

so inserting into (67) gives

$$\begin{aligned}
&\left\|\frac{d}{ds}\bar{f}(t,x)\right\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} \\
&\leq \|\bar{g}_1(t,x)\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} \\
&\quad +c\,\|\bar{g}_2(t,x)\|_{C^{r_0-1}(Q,\mathscr{L}(Z,Z_1))}\left\|\frac{dx}{ds}\right\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))} \\
&\leq \kappa_1(R)+c\kappa_1(R)R.
\end{aligned}$$

Combining this with (70), we obtain

$$\left\|\bar{f}(t,x)\right\|_{C^{r_0}(Q,Z_1)} \leq \kappa_1(R)(cR+1).$$

Furthermore, by (68),

$$\left\|\frac{dx}{ds}-\frac{dy}{ds}\right\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))} \leq \|x-y\|_{C^{r_0}(Q,Z)}. \tag{72}$$

Let $t_1, t_2 \in [a,b]$ and set

$$M_0 \;=\; \kappa_1(R)\left(|t_1-t_2|^{\varrho}+\|x-y\|^{\varrho}_{C^{r_0-1}(Q,Z)}\right). \tag{73}$$

Then (64–66) imply

$$\left\|\bar{f}(t_1,x)-\bar{f}(t_2,y)\right\|_{C^{r_0-1}(Q,Z_1)} \leq M_0 \tag{74}$$

$$\|\bar{g}_1(t_1,x)-\bar{g}_1(t_2,y)\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} \leq M_0 \tag{75}$$

$$\|\bar{g}_2(t_1,x)-\bar{g}_2(t_2,y)\|_{C^{r_0-1}(Q,\mathscr{L}(Z,Z_1))} \leq M_0. \tag{76}$$

Using (67), (69), (71–72), and (75–76) it follows that

$$\begin{aligned}
&\left\|\frac{d}{ds}\bar{f}(t_1,x)-\frac{d}{ds}\bar{f}(t_2,y)\right\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} \\
&\leq \|\bar{g}_1(t_1,x)-\bar{g}_1(t_2,y)\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z_1))} \\
&\quad +c\,\|\bar{g}_2(t_1,x)-\bar{g}_2(t_2,y)\|_{C^{r_0-1}(Q,\mathscr{L}(Z,Z_1))}\left\|\frac{dx}{ds}\right\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))} \\
&\quad +c\,\|\bar{g}_2(t_2,y)\|_{C^{r_0-1}(Q,\mathscr{L}(Z,Z_1))}\left\|\frac{dx}{ds}-\frac{dy}{ds}\right\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))} \\
&\leq (1+cR)M_0+c\kappa_1(R)\|x-y\|_{C^{r_0}(Q,Z)}.
\end{aligned}$$

Together with (73–74) this gives

$$\begin{aligned}
&\left\|\bar{f}(t_1,x)-\bar{f}(t_2,y)\right\|_{C^{r_0}(Q,Z_1)} \\
&\leq (1+cR)\kappa_1(R)\left(|t_1-t_2|^{\varrho}+\|x-y\|^{\varrho}_{C^{r_0-1}(Q,Z)}\right) \\
&\quad +c\kappa_1(R)\|x-y\|_{C^{r_0}(Q,Z)}.
\end{aligned}$$

Taking into account that by (68)

$$\|x-y\|_{C^{r_0}(Q,Z)} \le (2R)^{1-\varrho}\|x-y\|^{\varrho}_{C^{r_0}(Q,Z)},$$

this proves ϱ-Hölder continuity and thus (62). To prove (63) for r_0, it remains to show the Lipschitz property. This is analogous to the previous argument and we omit it here.

□

Lemma 3.3. *Given κ, L, there are $\kappa_1, L_1 : (0,+\infty) \to (0,+\infty)$ such that for all $f \in \mathscr{C}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa)$*

$$\bar{f} \in \mathcal{C}^{r,\varrho}([a,b] \times C^{r_0}(Q,Z), C^{r_0}(Q,Z_1); \kappa_1) \tag{77}$$

and for all $f \in \mathscr{C}^{r_0,r,\varrho}_{\mathrm{Lip}}(Q \times [a,b] \times Z, Z_1; \kappa, L)$

$$\bar{f} \in \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a,b] \times C^{r_0}(Q,Z), C^{r_0}(Q,Z_1); \kappa_1, L_1). \tag{78}$$

Proof. First we show (77). We argue by induction over r. The case $r = 0$ follows from (62) of Lemma 3.2. Now let $r \ge 1$ and assume that the statement holds for $r-1$. It follows from (48–51) that

$$\begin{aligned} f &\in \mathscr{C}^{r_0,r-1,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa) \\ g_1 := \frac{\partial f}{\partial t} &\in \mathscr{C}^{r_0,r-1,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa) \\ g_2 := \frac{\partial f}{\partial z} &\in \mathscr{C}^{r_0,r-1,\varrho}(Q \times [a,b] \times Z, \mathscr{L}(Z,Z_1); \kappa). \end{aligned}$$

The induction assumption implies

$$\bar{f} \in \mathcal{C}^{r-1,\varrho}([a,b] \times C^{r_0}(Q,Z), C^{r_0}(Q,Z_1); \kappa_1) \tag{79}$$

$$\bar{g}_1 \in \mathcal{C}^{r-1,\varrho}([a,b] \times C^{r_0}(Q,Z), C^{r_0}(Q,Z_1); \kappa_1) \tag{80}$$

$$\bar{g}_2 \in \mathcal{C}^{r-1,\varrho}([a,b] \times C^{r_0}(Q,Z), C^{r_0}(Q,\mathscr{L}(Z,Z_1)); \kappa_1). \tag{81}$$

Now we study the differentiability of $\bar{f}$ with respect to t and x, as a function from $[a,b] \times C^{r_0}(Q,Z)$ to $C^{r_0}(Q,Z_1)$. Let $t_1, t_2 \in [a,b]$, $t_1 \ne t_2$, $x \in C^{r_0}(Q,Z)$, $s \in Q$. Then

$$\begin{aligned} \frac{\bar{f}(t_2,x)(s) - \bar{f}(t_1,x)(s)}{t_2 - t_1} &= \frac{f(s,t_2,x(s)) - f(s,t_1,x(s))}{t_2 - t_1} \\ = \int_0^1 g_1(s, t_1 + \tau(t_2 - t_1), x(s))d\tau &= \int_0^1 \bar{g}_1(t_1 + \tau(t_2 - t_1), x)(s)d\tau. \end{aligned}$$

By (80), $\bar{g}_1$ is a continuous function from $[a,b] \times C^{r_0}(Q,Z)$ to $C^{r_0}(Q,Z_1)$, therefore, with the integral below considered in $C^{r_0}(Q,Z_1)$,

$$\frac{\bar{f}(t_2,x) - \bar{f}(t_1,x)}{t_2 - t_1} = \int_0^1 \bar{g}_1(t_1 + \tau(t_2 - t_1), x)d\tau$$

and

$$\lim_{t_2\to t_1} \sup_{\tau\in[0,1]} \|\bar g_1(t_1+\tau(t_2-t_1),x)-\bar g_1(t_1,x)\|_{C^{r_0}(Q,Z_1)}=0.$$

Consequently, with differentiation meant in $C^{r_0}(Q,Z_1)$,

$$\frac{\partial \bar f}{\partial t}=\bar g_1. \tag{82}$$

We introduce the following mapping

$$V: C^{r_0}(Q,\mathscr{L}(Z,Z_1))\to \mathscr{L}(C^{r_0}(Q,Z),C^{r_0}(Q,Z_1))$$

given for $w\in C^{r_0}(Q,\mathscr{L}(Z,Z_1))$, $x\in C^{r_0}(Q,Z)$, and $s\in Q$ by

$$((Vw)x)(s)=w(s)x(s).$$

Clearly, V is a bounded linear operator. This together with (81) yields

$$V\circ \bar g_2\in \mathcal{C}^{r-1,\varrho}\big([a,b]\times C^{r_0}(Q,Z),\mathscr{L}(C^{r_0}(Q,Z),C^{r_0}(Q,Z_1));\|V\|\kappa_1\big). \tag{83}$$

Next let $t\in[a,b]$, $x,y\in C^{r_0}(Q,Z)$, $\theta\in\mathbb{R}$, $\theta\neq 0$, $s\in Q$. Then we have

$$\begin{aligned}\frac{\bar f(t,x+\theta y)(s)-\bar f(t,x)(s)}{\theta}&=\frac{f(s,t,x(s)+\theta y(s))-f(s,t,x(s))}{\theta}\\ =\int_0^1 g_2(s,t,x(s)+\tau\theta y(s))y(s)d\tau&=\int_0^1 \big((V\bar g_2(t,x+\tau\theta y))y\big)(s)d\tau.\end{aligned}$$

Relation (83) shows that $V\circ\bar g_2$ is a continuous function from $[a,b]\times C^{r_0}(Q,Z)$ to $\mathscr{L}(C^{r_0}(Q,Z),C^{r_0}(Q,Z_1))$. It follows that

$$\frac{\bar f(t,x+\theta y)-\bar f(t,x)}{\theta} \;=\; \int_0^1 (V\bar g_2(t,x+\tau\theta y))y\,d\tau, \tag{84}$$

moreover,

$$\lim_{\theta\to 0}\sup_{\tau\in[0,1],y\in B_{C^{r_0}(Q,Z)}} \|V\bar g_2(t,x+\tau\theta y)-V\bar g_2(t,x)\|_{\mathscr{L}(C^{r_0}(Q,Z),C^{r_0}(Q,Z_1))}=0,$$

and hence

$$\lim_{\theta\to 0}\sup_{\tau\in[0,1],y\in B_{C^{r_0}(Q,Z)}} \|(V\bar g_2(t,x+\tau\theta y))y-(V\bar g_2(t,x))y\|_{C^{r_0}(Q,Z_1)}=0. \tag{85}$$

From (84) and (85) we conclude that $\bar f$ is Fréchet differentiable with respect to x as a function from $[a,b]\times C^{r_0}(Q,Z)$ to $C^{r_0}(Q,Z_1)$ and

$$\frac{\partial \bar f}{\partial x}=V\circ\bar g_2. \tag{86}$$

Combining (79–80), (82–83), and (86) completes the induction and thus the proof of (77). By (52),

$$\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa, L) \subseteq \mathscr{C}_{\mathrm{Lip}}^{r_0,0,\varrho}(Q \times [a,b] \times Z, Z_1; \kappa, L).$$

Therefore relation (63) of Lemma 3.2 yields the required Lipschitz property, which proves (78).

□

Given $(f, u_0) \in \mathscr{F}$, we recall that we consider the solution $u = u(s,t)$ of (55–56) also as a function $u(t) = u(\,\cdot\,, t)$ in $B([a,b], B(Q,Z))$, the required boundedness being a consequence of (58).

Lemma 3.4. *There is a constant $\lambda_1 > 0$ such that for all $(f, u_0) \in \mathscr{F}$ the following hold: $u(t) \in C^{r_0}(Q,Z)$ $(t \in [a,b])$, u is the unique solution of*

$$\frac{du(t)}{dt} \;=\; \bar{f}(t, u(t)) \quad (t \in [a,b]), \quad u(a) = u_0, \tag{87}$$

considered as an equation in $C^{r_0}(Q,Z)$, moreover,

$$\|u\|_{B([a,b],C^{r_0}(Q,Z))} \;\leq\; \lambda_1, \tag{88}$$

$$\|A^0_{n,\omega}(\bar{f}, u_0)\|_{B([a,b],C^{r_0}(Q,Z))} \;\leq\; \lambda_1 \quad (n \in \mathbb{N}, \omega \in \Omega). \tag{89}$$

Proof. Let $(f, u_0) \in \mathscr{F}$. We start with a preliminary argument. By Lemma 3.3,

$$\bar{f} \;\in\; \mathcal{C}_{\mathrm{Lip}}^{r_1,\varrho_1}([a,b] \times C^{r_0}(Q,Z), C^{r_0}(Q,Z); \kappa_1, L_1). \tag{90}$$

It follows that there exists a solution $w(t)$ of

$$\frac{dw(t)}{dt} \;=\; \bar{f}(t, w(t)) \quad (t \in [a, b_1)), \qquad w(a) = u_0, \tag{91}$$

considered as an ODE in $C^{r_0}(Q,Z)$, on a maximal interval $[a, b_1)$ with $a < b_1 \leq b$. Applying δ_s to (91) we get

$$\frac{d}{dt}(w(t), \delta_s) = \left(\frac{d}{dt} w(t), \delta_s\right) = (\bar{f}(t, w(t)), \delta_s) = f(s, t, (w(t), \delta_s)) \quad (t \in [a, b_1))$$

and

$$(w(a), \delta_s) = (u_0, \delta_s) = u_0(s).$$

By uniqueness of the solution to (55–56), we conclude

$$(w(t), \delta_s) = u(s,t) = (u(t), \delta_s) \quad (s \in Q, t \in [a, b_1)),$$

hence

$$w(t) = u(t) \quad (t \in [a, b_1)). \tag{92}$$

Now assume that

$$\sup_{t\in[a,b_1)} \|w(t)\|_{C^{r_0}(Q,Z)} := R_0 < \infty. \tag{93}$$

Then (90) implies that for all $t \in [a,b]$, $x, y \in (R_0+1)B_{C^{r_0}(Q,Z)}$

$$\begin{aligned} \|\bar{f}(t,x)\|_{C^{r_0}(Q,Z)} &\leq \kappa_1(R_0+1) \\ \|\bar{f}(t,x) - \bar{f}(t,y)\|_{C^{r_0}(Q,Z)} &\leq L_1(R_0+1)\|x-y\|_{C^{r_0}(Q,Z)}. \end{aligned}$$

Consequently, there is a $\delta > 0$ such that for any $b_2 \in [a, b_1)$ the solution $w(t)$ of (91) on $[a, b_2]$ can be continued to a solution on $[a, \min(b_2+\delta, b)]$ (see, e.g., [1], Ch. 2, Cor. 1.7.2, or use Banach's fix point theorem). It follows that $b_1 = b$ and $w(t)$ can be continued to a solution of (91) on $[a,b]$, that is,

$$w \in C^1([a,b], C^{r_0}(Q,Z)) \tag{94}$$

and

$$\frac{dw(t)}{dt} = \bar{f}(t, w(t)) \quad (t \in [a,b]), \qquad w(a) = u_0. \tag{95}$$

Since $u(s,\,\cdot\,) \in C^1([a,b],Z)$ $(s \in Q)$, we use continuity to conclude from (92) and (94) that

$$w(t) = u(t) \quad (t \in [a,b]) \tag{96}$$

and

$$\sup_{t\in[a,b]} \|u(t)\|_{C^{r_0}(Q,Z)} \leq R_0. \tag{97}$$

To summarize, so far we showed that (93) implies (94–97).

After this preparation we prove the lemma. We argue by induction over r_0. Let $r_0 = 0$. By (58) of the definition of $\mathscr{F}$ we have

$$\sup_{t\in[a,b_1)} \|u(t)\|_{C(Q,Z)} = \sup_{s\in Q, t\in[a,b_1)} \|u(s,t)\|_Z \leq \lambda.$$

Therefore (93) holds with $R_0 = \lambda$, so (96) and (97) imply (88) for $r_0 = 0$. Moreover, if $r = \varrho = r_1 = \varrho_1 = 0$, then (89) follows by (59), while for $r + \varrho > 0$ or $r_1 + \varrho_1 > 0$ we note that by (51) and (57)

$$\begin{aligned} f &\in \mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q \times [a,b] \times Z, Z; \kappa, L) \\ f &\in \mathscr{C}^{0,r_1,\varrho_1}_{\mathrm{Lip}}(Q \times [a,b] \times Z, Z; \kappa, L) \end{aligned}$$

and therefore, by Lemma 3.3

$$\begin{aligned} \bar{f} &\in \mathcal{C}^{r,\varrho}_{\mathrm{Lip}}([a,b] \times C(Q,Z), C(Q,Z); \kappa_1, L_1) \\ \bar{f} &\in \mathcal{C}^{r_1,\varrho_1}_{\mathrm{Lip}}([a,b] \times C(Q,Z), C(Q,Z); \kappa_1, L_1). \end{aligned}$$

Now (89) is a consequence of (the already proved) relation (88) for $r_0 = 0$ and Proposition 2.1 (for $n < \nu_0$ it follows directly from the boundedness properties of f and u_0).

Next let $r_0 \geq 1$ and assume the statements are true for $r_0 - 1$. Let $(f, u_0) \in \mathscr{F}$ and put

$$\begin{aligned} g_1 &:= \frac{\partial f}{\partial s} \in \mathscr{C}^{r_0-1,r,\varrho}(Q \times [a,b] \times Z, \mathscr{L}(\mathbb{R}^{d_0}, Z); \kappa) && (98) \\ g_2 &:= \frac{\partial f}{\partial z} \in \mathscr{C}^{r_0-1,r,\varrho}(Q \times [a,b] \times Z, \mathscr{L}(Z, Z); \kappa). && (99) \end{aligned}$$

By Lemma 3.3,

$$\begin{aligned} \bar{g}_1 &\in \mathcal{C}^{r,\varrho}([a,b] \times C^{r_0-1}(Q,Z), C^{r_0-1}(Q, \mathscr{L}(\mathbb{R}^{d_0}, Z)); \kappa_1) && (100) \\ \bar{g}_2 &\in \mathcal{C}^{r,\varrho}([a,b] \times C^{r_0}(Q,Z), C^{r_0}(Q, \mathscr{L}(Z, Z)); \kappa_1). && (101) \end{aligned}$$

We start with the proof of (88). By the induction assumption, $u(t)$ is the solution of (87), considered in $C^{r_0-1}(Q,Z))$, and

$$\|u\|_{B([a,b],C^{r_0-1}(Q,Z))} \leq c_0. \qquad (102)$$

From (100–102) and the assumptions on u_0 we conclude that there is a $c_1 > 0$ such that

$$\begin{aligned} \sup_{t\in[a,b]} \|\bar{g}_1(t,u(t))\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))} &\leq c_1 && (103) \\ \sup_{t\in[a,b]} \|\bar{g}_2(t,u(t))\|_{C^{r_0-1}(Q,\mathscr{L}(Z))} &\leq c_1 && (104) \\ \left\| \frac{du_0}{ds} \right\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))} &\leq \sigma. && (105) \end{aligned}$$

By uniqueness, $w(t) = u(t)$, where w is the solution of (91), so $u(t) \in C^{r_0}(Q,Z)$ for all $t \in [a, b_1)$ and $u(t)$ is continuously differentiable as a function from $[a, b_1)$ to $C^{r_0}(Q,Z)$. Let D be differentiation $\frac{d}{ds}$, considered as an operator $D \in \mathscr{L}(C^{r_0}(Q,Z), C^{r_0-1}(Q, \mathscr{L}(\mathbb{R}^{d_0}, Z)))$. Then applying D to (91) with $w = u$ and inserting (98–99), we get

$$\begin{aligned} \frac{d(Du(t))}{dt} &= D\frac{du(t)}{dt} = D\bar{f}(t,u(t)) \\ &= \bar{g}_1(t,u(t)) + \bar{g}_2(t,u(t))Du(t) \quad (t \in [a,b_1)) \\ Du(a) &= Du_0. \end{aligned}$$

Integrating with respect to t, we obtain for $t \in [a, b_1)$

$$Du(t) = Du_0 + \int_a^t (\bar{g}_1(\tau,u(\tau)) + \bar{g}_2(\tau,u(\tau))Du(\tau))\, d\tau.$$

Using (103–105) we conclude for $t \in [a, b_1)$

$$
\begin{aligned}
&\|Du(t)\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))} \\
&\quad\le \|Du_0\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))} + \int_a^t \|\bar{g}_1(\tau, u(\tau))\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))}\, d\tau \\
&\qquad + \int_a^t c_2 \|\bar{g}_2(\tau, u(\tau))\|_{C^{r_0-1}(Q,\mathscr{L}(Z))} \|Du(\tau)\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))}\, d\tau \\
&\quad\le \sigma + (b-a)c_1 + c_1 c_2 \int_a^t \|Du(\tau)\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))}\, d\tau.
\end{aligned}
$$

Since $t \to Du(t)$ is a continuous function from $[a, b_1)$ to $C^{r_0-1}(Q, \mathscr{L}(\mathbb{R}^{d_0}, Z))$, we can use Gronwall's lemma to get

$$
\sup_{t\in[a,b_1)} \|Du(t)\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))} \le (\sigma + (b-a)c_1)e^{c_1 c_2 (b-a)} := c_3,
$$

which together with (102) gives

$$
\sup_{t\in[a,b_1)} \|w(t)\|_{C^{r_0}(Q,Z)} = \sup_{t\in[a,b_1)} \|u(t)\|_{C^{r_0}(Q,Z)} \le \max(c_0, c_3) := c_4. \tag{106}
$$

Consequently, (93) holds with $R_0 = c_4$, so (96) and (97) give (88) for r_0.

Now we turn to (89). By (11–14),

$$
A^0_{n,\omega}(\bar{f}, u_0) = v \in B([a,b], C(Q,Z)),
$$

where

$$
v(t) = \begin{cases} p_{k,0}(t) & \text{if} \quad t \in [t_k, t_{k+1}) \quad \text{and} \quad 0 \le k \le n-1 \\ u_n & \text{if} \quad t = t_n \end{cases}
$$

and for $k = 0, \ldots, n-1$

$$
p_{k,0}(t) = u_k + \bar{f}(t_k, u_k)(t - t_k) \quad (t \in [t_k, t_{k+1}]) \tag{107}
$$

$$
u_{k+1} = u_k + h\bar{f}(\xi_{k+1}, p_{k,0}(\xi_{k+1})). \tag{108}
$$

The induction assumption implies

$$
\max_{0\le k\le n-1} \max_{t\in[t_k,t_{k+1}]} \|p_{k,0}(t)\|_{C^{r_0-1}(Q,Z)} \le c_0 \tag{109}
$$

and

$$
\max_{0\le k\le n} \|u_k\|_{C^{r_0-1}(Q,Z)} \le c_0. \tag{110}
$$

Using (90) and $u_0 \in C^{r_0}(Q,Z)$, it readily follows from (107–108) that for $0 \le k \le n-1$

$$
p_{k,0}(t) \in C^{r_0}(Q,Z) \quad (t \in [t_k, t_{k+1}],\ 0 \le k \le n-1) \tag{111}
$$

$$
u_k \in C^{r_0}(Q,Z) \quad (0 \le k \le n). \tag{112}
$$

Differentiating (107) and (108), we obtain for $0 \leq k \leq n-1$

$$\frac{dp_{k,0}(\xi_{k+1})}{ds} = \frac{du_k}{ds} + (\xi_{k+1} - t_k)\bar{g}_1(t_k, u_k) + (\xi_{k+1} - t_k)\bar{g}_2(t_k, u_k)\frac{du_k}{ds} \quad (113)$$

$$\begin{aligned} \frac{du_{k+1}}{ds} &= \frac{du_k}{ds} + h\bar{g}_1\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big) \\ &\quad + h\bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big)\frac{dp_{k,0}(\xi_{k+1})}{ds}. \end{aligned} \quad (114)$$

Inserting (113) into (114), we get

$$\begin{aligned} \frac{du_{k+1}}{ds} &= \frac{du_k}{ds} + h\bar{g}_1\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big) + h\bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big)\frac{du_k}{ds} \\ &\quad + h(\xi_{k+1} - t_k)\bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big)\bar{g}_1(t_k, u_k) \\ &\quad + h(\xi_{k+1} - t_k)\bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big)\bar{g}_2(t_k, u_k)\frac{du_k}{ds}. \end{aligned}$$

Hence,

$$\frac{du_{k+1}}{ds} = (I_Z + hv_k)\frac{du_k}{ds} + hw_k, \quad (115)$$

where

$$\begin{aligned} v_k &= \bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big) + (\xi_{k+1} - t_k)\bar{g}_2\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big)\bar{g}_2(t_k, u_k) \\ w_k &= \bar{g}_1\big(\xi_{k+1}, p_{k,0}(\xi_{k+1})\big) + (\xi_{k+1} - t_k)\bar{g}_2\big(\xi_{k+1}, p_{k,0}(s, \xi_{k+1})\big)\bar{g}_1(t_k, u_k). \end{aligned}$$

By (100–101) and (109–110) it follows that

$$\max_{0 \leq k \leq n-1} \|v_k\|_{C^{r_0-1}(Q, \mathscr{L}(Z))} \leq c_1 \quad (116)$$

$$\max_{0 \leq k \leq n-1} \|w_k\|_{C^{r_0-1}(Q, \mathscr{L}(\mathbb{R}^{d_0}, Z))} \leq c_1. \quad (117)$$

Moreover, by the assumption on u_0,

$$\left\|\frac{du_0}{ds}\right\|_{C^{r_0-1}(Q, \mathscr{L}(\mathbb{R}^{d_0}, Z))} \leq \sigma. \quad (118)$$

We have by (115–117)

$$\begin{aligned} &\left\|\frac{du_{k+1}}{ds}\right\|_{C^{r_0-1}(Q, \mathscr{L}(\mathbb{R}^{d_0}, Z))} \\ &\leq \Big(1 + c_2 h \|v_k\|_{C^{r_0-1}(Q, \mathscr{L}(Z))}\Big) \left\|\frac{du_k}{ds}\right\|_{C^{r_0-1}(Q, \mathscr{L}(\mathbb{R}^{d_0}, Z))} + h\|w_k\|_{C^{r_0-1}(Q, \mathscr{L}(\mathbb{R}^{d_0}, Z))} \\ &\leq (1 + c_1 c_2 h)\left\|\frac{du_k}{ds}\right\|_{C^{r_0-1}(Q, \mathscr{L}(\mathbb{R}^{d_0}, Z))} + c_1 h. \end{aligned}$$

From this and (118) we conclude for $1 \le k \le n$

$$
\begin{aligned}
\left\| \frac{du_k}{ds} \right\|_{C^{r_0-1}(Q,\mathscr{L}(\mathbb{R}^{d_0},Z))} &\le \sigma(1+c_1c_2h)^k + c_1h\sum_{j=0}^{k-1}(1+c_1c_2h)^j \\
&\le \sigma(1+c_1c_2h)^k + c_1h\frac{(1+c_1c_2h)^k-1}{c_1c_2h} \\
&\le \sigma(1+1/c_2)(1+c_1c_2h)^n \le \sigma(1+1/c_2)e^{c_1c_2nh} \\
&= \sigma(1+1/c_2)e^{c_1c_2(b-a)} := c_3.
\end{aligned}
$$

Combining this with (110) gives

$$
\max_{0\le k\le n} \|u_k\|_{C^{r_0}(Q,Z)} \le c_4 := \max(c_0, c_3),
$$

which, taking into account (107) and (90), also yields

$$
\max_{0\le k\le n-1} \max_{t\in[t_k,t_{k+1}]} \|p_{k,0}(t)\|_{C^{r_0}(Q,Z)} \le c_5,
$$

and hence the desired result.

□

Proof of Proposition 3.1. The result follows from Lemmas 3.3–3.4, taking into account that (88) and (89) for $r_0 > 0$ imply the respective estimates also for $r_0 = 0$.

4 The algorithm and its analysis

For $l \in \mathbb{N}_0$ let Γ_l be the equidistant grid on Q of meshsize $(\max(r_0,1))^{-1}2^{-l}$ and let $\{Q_{li} : i = 1, 2, \dots, 2^{d_0 l}\}$ be the partition of Q into cubes of sidelength 2^{-l}. Define the following operators E_{li} and R_{li} acting on $\Phi(\mathbb{R}^{d_0}, Z)$, the space of all functions from $\mathbb{R}^{d_0}$ to Z: For $f \in \Phi(\mathbb{R}^{d_0}, Z)$ and $s \in \mathbb{R}^{d_0}$ put

$$
\begin{aligned}
(E_{li}f)(s) &= f(s_{li} + 2^{-l}s) && (119)\\
(R_{li}f)(s) &= f(2^l(s - s_{li})), && (120)
\end{aligned}
$$

where s_{li} is the point in Q_{li} with minimal coordinates. We also apply these operators to functions which are defined on subsets of $\mathbb{R}^{d_0}$. In this case we assume that the function is extended to $\mathbb{R}^{d_0}$ by zero. Let for $f \in \Phi(\mathbb{R}^{d_0}, Z)$

$$
Pf = \sum_{j=1}^{\nu_1} f(a_j)\varphi_j \tag{121}
$$

be the Z-valued tensor product Lagrange interpolation operator of degree $\max(r_0, 1)$, where $(a_j)_{j=1}^{\nu_1}$ are the points of Γ_0 and $(\varphi_j)_{j=1}^{\nu_1}$ are the respective scalar Lagrange polynomials, considered as functions on $\mathbb{R}^{d_0}$. If $\mathcal{P}_{\max(r_0,1)}$ denotes the space of polynomials on $\mathbb{R}^{d_0}$ of degree at most $\max(r_0, 1)$, with coefficients in Z, then we have

$$Pg = g \quad (g \in \mathcal{P}_{\max(r_0,1)}).$$

Define $P_l : \Phi(Q, Z) \to C(Q, Z)$ for $l \in \mathbb{N}_0$ by

$$(P_l f)(s) = (R_{li} P E_{li} f)(s) \quad (s \in Q_{li}, f \in \Phi(Q, Z)),$$

thus, by (121),

$$(P_l f)(s) = \sum_{j=1}^{\nu_1} f(s_{li} + 2^{-l} a_j) \varphi_j(2^l(s - s_{li})) \quad (s \in Q_{li}),$$

so P_l is Z-valued composite with respect to the partition Q_{li} tensor product Lagrange interpolation of degree $\max(r_0, 1)$. Hence,

$$(P_l f)(s) = f(s) \quad (s \in \Gamma_l, f \in \Phi(Q, Z))$$

and P_l is of the form

$$P_l f = \sum_{s \in \Gamma_l} f(s) \psi_{ls} \quad (f \in \Phi(Q, Z)) \tag{122}$$

with $\psi_{ls} \in C(Q)$.

Let $(f, u_0) \in \mathscr{F}$. We define the following multilevel algorithm for the approximate solution of the parametric problem (55–56). Let $l_0, l_1 \in \mathbb{N}_0$, $l_0 \le l_1$, $n_{l_0}, \ldots, n_{l_1} \in \mathbb{N}$, $\omega \in \Omega$, and set

$$\begin{aligned} \mathscr{A}_\omega(f, u_0) \;&=\; P_{l_0}\left(\left(A^r_{n_{l_0},\omega}(f_s, u_0(s))\right)_{s \in \Gamma_{l_0}}\right) \\ &\quad + \sum_{l=l_0+1}^{l_1} (P_l - P_{l-1})\left(\left(A^{r_1}_{n_l,\omega}(f_s, u_0(s))\right)_{s \in \Gamma_l}\right), \end{aligned} \tag{123}$$

where we use the respective algorithms given by (7–12). Let $\operatorname{card}(\mathscr{A}_\omega)$ denote the number of function evaluations involved in $\mathscr{A}_\omega$. We have

$$\operatorname{card}(\mathscr{A}_\omega) \le c \sum_{l=l_0}^{l_1} n_l 2^{d_0 l}. \tag{124}$$

Note also that the number of arithmetic operations of $\mathscr{A}_\omega$ (including additions in Z and multiplications of elements of Z by scalars) is bounded from above by $c \operatorname{card}(\mathscr{A}_\omega)$ for some $c > 0$.

Theorem 4.1. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $0 \le \varrho, \varrho_1 \le 1$ with $r+\varrho \ge r_1+\varrho_1$, let $\kappa, L : (0,+\infty) \to (0,+\infty)$, $\sigma, \lambda > 0$, and $1 \le p \le 2$. Then there are constants $c_1, c_2 > 0$ and $\nu_0 \in \mathbb{N}$ such that the following holds. Let Z be a Banach space and let $\mathscr{F}$ be defined by (57). For all $l_0, l_1 \in \mathbb{N}_0$ with $l_0 \le l_1$ and for all $(n_l)_{l=l_0}^{l_1} \subset \mathbb{N}$ with $n_l \ge \nu_0$ $(l_0 \le l \le l_1)$ the so-defined algorithm $(\mathscr{A}_\omega)_{\omega\in\Omega}$ satisfies*

$$\begin{aligned}&\sup_{(f,u_0)\in\mathscr{F}} \|\mathscr{S}(f,u_0)-\mathscr{A}_\omega(f,u_0)\|_{B(Q\times[a,b],Z)}\\ &\le\ c_1 2^{-r_0 l_1} + c_1 n_{l_0}^{-r-\varrho} + c_1 \sum_{l=l_0+1}^{l_1} 2^{-r_0 l} n_l^{-r_1-\varrho_1} \quad (\omega\in\Omega) \qquad (125)\end{aligned}$$

and for all l^* with $l_0 \le l^* \le l_1$

$$\begin{aligned}&\sup_{(f,u_0)\in\mathscr{F}} \left(\mathbb{E}\,\|\mathscr{S}(f,u_0)-\mathscr{A}_\omega(f,u_0)\|_{B(Q\times[a,b],Z)}^p\right)^{1/p}\\ &\le\ c_2 2^{-r_0 l_1} + c_2 (l_0+1)^{1/2}\tau_p(Z) n_{l_0}^{-r-\varrho-1+1/p}\\ &\quad + c_2 \sum_{l=l_0+1}^{l^*} (l+1)^{1/2}\tau_p(Z) 2^{-r_0 l} n_l^{-r_1-\varrho_1-1+1/p} + c_2 \sum_{l=l^*+1}^{l_1} 2^{-r_0 l} n_l^{-r_1-\varrho_1}. \qquad (126)\end{aligned}$$

Remark 4.2. Observe that the restriction $r+\varrho \ge r_1+\varrho_1$ in Theorem 4.1 is no loss of generality. Indeed, if $r+\varrho < r_1+\varrho_1$, then either $r < r_1$ or $(r = r_1) \wedge (\varrho < \varrho_1)$. It follows from (52–54) that in both cases we have

$$\mathscr{C}_{\mathrm{Lip}}^{r_0,r_1,\varrho_1}(Q\times[a,b]\times Z, Z;\kappa, L)\ \subseteq\ \mathscr{C}_{\mathrm{Lip}}^{0,r,\varrho}(Q\times[a,b]\times Z, Z; 2\kappa, L).$$

Consequently,

$$\begin{aligned}&\mathscr{C}_{\mathrm{Lip}}^{r_0,r_1,\varrho_1}(Q\times[a,b]\times Z, Z;\kappa/2, L)\\ &\subseteq\ \mathscr{C}_{\mathrm{Lip}}^{0,r,\varrho}(Q\times[a,b]\times Z, Z;\kappa, L)\cap \mathscr{C}_{\mathrm{Lip}}^{r_0,r_1,\varrho_1}(Q\times[a,b]\times Z, Z;\kappa, L)\\ &\subseteq\ \mathscr{C}_{\mathrm{Lip}}^{r_0,r_1,\varrho_1}(Q\times[a,b]\times Z, Z;\kappa, L),\end{aligned}$$

which by (52) and (57) means that the case $r+\varrho < r_1+\varrho_1$ is essentially the same as the case $r = r_1$, $\varrho = \varrho_1$.

For the reason to consider a variable summation index l^* in (126) we refer to Remark 2.4.

Corollary 4.3. *Assume the conditions of Theorem 4.1 and let Z be of type p. Then there are constants $c_{1-3} > 0$ such that for all $n \in \mathbb{N}$ with $n \ge 2$ the following hold. Setting*

$$l_1 = \left\lceil \frac{\log n}{d_0} \right\rceil, \quad l_0 = \left\lfloor \frac{r+\varrho-r_1-\varrho_1}{\frac{r_0}{d_0}+r+\varrho-r_1-\varrho_1} l_1 \right\rfloor, \qquad (127)$$

$$n_l = \nu_0 \left\lceil 2^{d_0(l_1-l)} \right\rceil \quad (l_0 \le l \le l_1), \qquad (128)$$

the so-defined algorithm $(\mathscr{A}_\omega)_{\omega\in\Omega}$ *fulfills*

$$\operatorname{card}(\mathscr{A}_\omega) \le c_1 n \log n \quad (\omega \in \Omega). \tag{129}$$

Moreover,

$$\sup_{(f,u_0)\in\mathscr{F}} \|\mathscr{S}(f,u_0) - \mathscr{A}_\omega(f,u_0)\|_{B(Q\times[a,b],Z)} \le c_2 n^{-\upsilon_1}(\log n)^{\theta_1} \quad (\omega \in \Omega), \tag{130}$$

where

$$\upsilon_1 = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+r+\varrho-r_1-\varrho_1}(r+\varrho) & \text{if } \frac{r_0}{d_0} > r_1+\varrho_1 \\ \frac{r_0}{d_0} & \text{if } \frac{r_0}{d_0} \le r_1+\varrho_1 \end{cases} \tag{131}$$

and

$$\theta_1 = \begin{cases} 0 & \text{if } \frac{r_0}{d_0} \ne r_1+\varrho_1 \\ 1 & \text{if } \frac{r_0}{d_0} = r_1+\varrho_1. \end{cases}$$

Finally,

$$\sup_{(f,u_0)\in\mathscr{F}} \left(\mathbb{E}\,\|\mathscr{S}(f,u_0) - \mathscr{A}_\omega(f,u_0)\|^p_{B(Q\times[a,b],Z)}\right)^{1/p} \le c_3 n^{-\upsilon_2(p)}(\log n)^{\theta_2(p)}, \tag{132}$$

with

$$\upsilon_2(p) = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+r+\varrho-r_1-\varrho_1}\left(r+\varrho+1-\frac{1}{p}\right) & \text{if } \frac{r_0}{d_0} > r_1+\varrho_1+1-\frac{1}{p} \\ \frac{r_0}{d_0} & \text{if } \frac{r_0}{d_0} \le r_1+\varrho_1+1-\frac{1}{p} \end{cases} \tag{133}$$

and

$$\theta_2(p) = \begin{cases} \frac{1}{2} & \text{if } \frac{r_0}{d_0} \ne r_1+\varrho_1+1-\frac{1}{p} \\ \frac{3}{2} & \text{if } \frac{r_0}{d_0} = r_1+\varrho_1+1-\frac{1}{p}. \end{cases}$$

First we derive Corollary 4.3 from Theorem 4.1.

Proof. Relation (129) follows directly from (124) and (128). Next observe that by (127)

$$l_1 - l_0 = \left\lceil \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+r+\varrho-r_1-\varrho_1} l_1 \right\rceil, \tag{134}$$

and therefore

$$\begin{aligned} n_{l_0}^{-r-\varrho} &\le 2^{-(r+\varrho)d_0(l_1-l_0)} = 2^{-(r+\varrho-r_1-\varrho_1)d_0(l_1-l_0)-(r_1+\varrho_1)d_0(l_1-l_0)} \\ &\le 2^{-r_0 l_0-(r_1+\varrho_1)d_0(l_1-l_0)}. \end{aligned} \tag{135}$$

Furthermore,

$$2^{-r_0 l} n_l^{-r_1-\varrho_1} \le 2^{-r_0 l-(r_1+\varrho_1)d_0(l_1-l)} \quad (l_0 < l \le l_1). \tag{136}$$

Now (125) together with (134–136) gives

$$\begin{aligned}
&\sup_{(f,u_0)\in\mathscr{F}} \|\mathscr{S}(f,u_0)-\mathscr{A}_\omega(f,u_0)\|_{B(Q\times[a,b],Z)} \\
&\le c\sum_{l=l_0}^{l_1} 2^{-r_0 l-(r_1+\varrho_1)d_0(l_1-l)} \\
&\le \begin{cases} c2^{-r_0 l_0-(r_1+\varrho_1)d_0(l_1-l_0)} \le cn^{-\upsilon_1} & \text{if } \frac{r_0}{d_0} > r_1+\varrho_1 \\ c(l_1-l_0+1)2^{-r_0 l_1} \le cn^{-r_0/d_0}\log n & \text{if } \frac{r_0}{d_0} = r_1+\varrho_1 \\ c2^{-r_0 l_1} \le cn^{-r_0/d_0} & \text{if } \frac{r_0}{d_0} < r_1+\varrho_1, \end{cases}
\end{aligned}$$

which proves (130). In a similar way, setting $\beta = r_1+\varrho_1+1-1/p$, (126) with $l^* = l_1$ and (134–136) yield

$$\begin{aligned}
&\sup_{(f,u_0)\in\mathscr{F}} \left(\mathbb{E}\,\|\mathscr{S}(f,u_0)-\mathscr{A}_\omega(f,u_0)\|^p_{B(Q\times[a,b],Z)}\right)^{1/p} \\
&\le c\sum_{l=l_0}^{l_1} (l+1)^{1/2} 2^{-r_0 l-\beta d_0(l_1-l)} \\
&\le \begin{cases} c(l_1+1)^{1/2}2^{-r_0 l_0-\beta d_0(l_1-l_0)} \le cn^{-\upsilon_2(p)}(\log n)^{1/2} & \text{if } \frac{r_0}{d_0} > \beta \\ c(l_1-l_0+1)(l_1+1)^{1/2}2^{-r_0 l_1} \le cn^{-r_0/d_0}(\log n)^{3/2} & \text{if } \frac{r_0}{d_0} = \beta \\ c(l_1+1)^{1/2}2^{-r_0 l_1} \le cn^{-r_0/d_0}(\log n)^{1/2} & \text{if } \frac{r_0}{d_0} < \beta, \end{cases}
\end{aligned}$$

which shows (132). □

Remark 4.4. Concerning Corollary 4.3 we note that balancing the n_l more cleverly could reduce the cost to $c_1 n$ in some regions of the smoothness parameters. However, this balancing could lead to further logarithmic factors in either the deterministic or the randomized setting. Since in view of Corollary 5.2 in general Banach spaces even the optimal exponent is known only up to an arbitrary small $\varepsilon > 0$, we neglect the aspect of improving the logarithms. See also the comment at the end of Section 6.

Also note that the choice of the parameters (127–128) depends only on the smoothness class, not on the setting. This means that the randomized algorithm satisfies the (usually stronger) error bound of the randomized setting, while each realization also satisfies the deterministic bound.

The proof of Theorem 4.1 will be given after some preparations. First we show that there are constants $c_1, c_2 > 0$ such that for all Banach spaces Z and $l \in \mathbb{N}_0$

$$\|P_l\|_{\mathscr{L}(C(Q,Z))} \le c_1 \tag{137}$$

$$\|J-P_lJ\|_{\mathscr{L}(C^{r_0}(Q,Z),C(Q,Z))} \le c_2 2^{-r_0 l}, \tag{138}$$

where $J : C^{r_0}(Q,Z) \to C(Q,Z)$ is the canonical embedding. This is well-known in the scalar case and easily extended to the Banach space case as follows. Denote by $P_l^{\mathbb{R}}$ and $J^{\mathbb{R}}$ the respective scalar operators. Then we have

$$\begin{aligned} \|P_l f\|_{C(Q,Z)} &= \sup_{z^* \in B_{Z^*}} \| (P_l f, z^*) \|_{C(Q)} = \sup_{z^* \in B_{Z^*}} \left\| P_l^{\mathbb{R}} (f, z^*) \right\|_{C(Q)} \\ &\le c_1 \sup_{z^* \in B_{Z^*}} \| (f, z^*) \|_{C(Q)} = c_1 \|f\|_{C(Q,Z)} \end{aligned} \tag{139}$$

and similarly,

$$\begin{aligned} &\|(J - P_l J) f\|_{C(Q,Z)} \\ &= \sup_{z^* \in B_{Z^*}} \left\| (J^{\mathbb{R}} - P_l^{\mathbb{R}} J^{\mathbb{R}}) (f, z^*) \right\|_{C(Q)} \le c_2 2^{-r_0 l} \sup_{z^* \in B_{Z^*}} \| (f, z^*) \|_{C^{r_0}(Q)} \\ &= c_2 2^{-r_0 l} \max_{0 \le j \le r_0} \sup_{z^* \in B_{Z^*}} \left\| \left(\frac{d^j f}{ds^j}, z^* \right) \right\|_{C(Q, \mathscr{L}_j(\mathbb{R}^{d_0}, \mathbb{R}))} \\ &= c_2 2^{-r_0 l} \|f\|_{C^{r_0}(Q,Z)}. \end{aligned} \tag{140}$$

In order to apply Proposition 2.3 we now construct operators $T_l : C(Q,Z) \to C^{r_0}(Q,Z)$ with certain suitable boundedness properties. Put

$$U = \left[-\frac{1}{\max(r_0,1)}, 1 + \frac{1}{\max(r_0,1)} \right]^{d_0},$$

$\mathcal{I}_l = \{1, 2, \dots, 2^{d_0 l}\}$, and for $l \in \mathbb{N}_0$, $i \in \mathcal{I}_l$

$$U_{li} = s_{li} + 2^{-l} U. \tag{141}$$

Let $\eta \in C^\infty(\mathbb{R}^{d_0})$ be such that $\eta \ge 0$, $\eta \equiv 1$ on Q, and $\operatorname{supp}(\eta) \subseteq U$. Then

$$\sum_{i \in \mathcal{I}_l} (R_{li} \eta)(s) \ge 1 \qquad (s \in Q, l \in \mathbb{N}_0). \tag{142}$$

Define functions η_{li} on Q $(i \in \mathcal{I}_l, l \in \mathbb{N}_0)$ by

$$\eta_{li}(s) = \frac{(R_{li}\eta)(s)}{\sum_{j \in \mathcal{I}_l} (R_{lj}\eta)(s)} \qquad (s \in Q). \tag{143}$$

We define $T_l : \Phi(Q,Z) \to C^{r_0}(Q,Z)$ for $l \in \mathbb{N}_0$ and $f \in \Phi(Q,Z)$ by

$$(T_l f)(s) = \sum_{i \in \mathcal{I}_l} \eta_{li}(s) \, (R_{li} P E_{li} f)(s) \quad (s \in Q), \tag{144}$$

consequently, using (119) and (121),

$$(T_l f)(s) = \sum_{i \in \mathcal{I}_l} \sum_{j=1}^{\nu_1} f(s_{li} + 2^{-l} a_j) \, \eta_{li}(s) \, R_{li} \varphi_j(s) \quad (s \in Q). \tag{145}$$

Thus, T_l is of the form

$$T_l f = \sum_{s\in\Gamma_l} f(s)\zeta_{ls} \quad (f \in \Phi(Q,Z)) \tag{146}$$

with $\zeta_{ls} \in C^{r_0}(Q)$.

For the proof of the next lemma we denote for $f \in C^m(Q)$

$$|f|_{m,Q} = \left\| \frac{d^m f}{ds^m} \right\|_{C(Q,\mathscr{L}_m(\mathbb{R}^{d_0},\mathbb{R}))}.$$

Lemma 4.5. *There are constants $c_1, c_2 > 0$ such that for all Banach spaces Z and $l \in \mathbb{N}_0$*

$$\|T_l\|_{\mathscr{L}(C(Q,Z))} \leq c_1 \tag{147}$$

$$\|T_l\|_{\mathscr{L}(C^{r_0}(Q,Z))} \leq c_2. \tag{148}$$

Moreover, for $f \in \Phi(Q,Z)$

$$(T_l f)(s) = f(s) \quad (s \in \Gamma_l). \tag{149}$$

Proof. We first prove the result for $Z = \mathbb{R}$. We have

$$\eta_{li}(s) \geq 0 \quad (s \in Q) \tag{150}$$

$$\eta_{li}(s) = 0 \quad (s \in Q \setminus U_{li}) \tag{151}$$

$$\sum_{i\in\mathcal{I}_l} \eta_{li}(s) = 1 \quad (s \in Q). \tag{152}$$

Moreover, there are constants $c_1, c_2 > 0$ such that for $m \in \mathbb{N}_0$, $0 \leq m \leq r_0$, $l \in \mathbb{N}_0$

$$\|R_{li}\eta\|_{C^m(\mathbb{R}^{d_0})} \leq c_1\, 2^{ml} \qquad (i \in \mathcal{I}_l) \tag{153}$$

and

$$\Big\| \sum_{i\in\mathcal{I}_l} R_{li}\eta \Big\|_{C^m(\mathbb{R}^{d_0})} \leq c_2\, 2^{ml}. \tag{154}$$

From (142–143) and (153–154) we get for $0 \leq m \leq r_0$

$$\|\eta_{li}\|_{C^m(Q)} \leq c\, 2^{ml}. \tag{155}$$

First we show (149). Let $s \in \Gamma_l$. If $s \in Q_{li}$, then $(R_{li}PE_{li}f)(s) = f(s)$. On the other hand, by the definition of the support of η, if $s \notin Q_{li}$, then $s \notin U_{li}^0$ (the interior of U_{li}), hence $(R_{li}\eta)(s) = 0$, and therefore $\eta_{li}(s) = 0$. This together with (144) and (152) implies (149).

Relation (147) is an immediate consequence of (144) and (150–152). Now we turn to (148). Due to (147), we can assume that $r_0 > 0$. By (121), for $f \in C^{r_0}(U)$ and $0 \leq m \leq r_0$

$$\|Pf\|_{C^m(U)} \leq c\|f\|_{C^{r_0}(U)},$$

and consequently,

$$\begin{aligned}\|f - Pf\|_{C^m(U)} &= \inf_{g\in\mathcal{P}_{r_0}} \|(f-g) - P(f-g)\|_{C^m(U)} \\ &\le c \inf_{g\in\mathcal{P}_{r_0}} \|f-g\|_{C^{r_0}(U)} \le c|f|_{r_0,U}, \end{aligned} \tag{156}$$

where the latter relation is an application of Theorem 3.1.1 from [2] (this theorem is formulated for Sobolev spaces $W^{r_0}_\infty(U)$, but since $f, Pf, g \in C^{r_0}(U)$, the corresponding (semi-)norms coincide). Let $f \in C^{r_0}(Q)$ and let $\tilde{f} \in C^{r_0}(\mathbb{R}^{d_0})$ be an extension of f with

$$\|\tilde{f}\|_{C^{r_0}(\mathbb{R}^{d_0})} \le c\|f\|_{C^{r_0}(Q)},$$

which exists due to the Whitney extension theorem, see [30], [18], Th. 2.3.6. From (151) and (152) we conclude

$$\begin{aligned}\|f - T_l f\|_{C^{r_0}(Q)} &= \Big\| \sum_{i\in\mathcal{I}_l} \eta_{li}(f - R_{li}PE_{li}f)\Big\|_{C^{r_0}(Q)} \\ &\le c \max_{i\in\mathcal{I}_l} \|\eta_{li}(f - R_{li}PE_{li}f)\|_{C^{r_0}(Q)}. \end{aligned} \tag{157}$$

Furthermore, for $0 \le m \le r_0$

$$\|R_{li}g\|_{C^m(U_{li})} \le c2^{ml}\|g\|_{C^m(U)} \quad (g \in C^m(U)) \tag{158}$$

and, using (155),

$$\|\eta_{li}g\|_{C^{r_0}(Q\cap U_{li})} \le c\sum_{m=0}^{r_0} 2^{(r_0-m)l}\|g\|_{C^m(Q\cap U_{li})} \quad (g \in C^{r_0}(Q\cap U_{li})). \tag{159}$$

Applying (158–159) and (156), we obtain

$$\begin{aligned}\|\eta_{li}(f - R_{li}PE_{li}f)\|_{C^{r_0}(Q)} &= \|\eta_{li}(f - R_{li}PE_{li}f)\|_{C^{r_0}(Q\cap U_{li})} \\ \le\ & c\sum_{m=0}^{r_0} 2^{(r_0-m)l}\|f - R_{li}PE_{li}f\|_{C^m(Q\cap U_{li})} \\ \le\ & c\sum_{m=0}^{r_0} 2^{(r_0-m)l}\|\tilde{f} - R_{li}PE_{li}\tilde{f}\|_{C^m(U_{li})} \\ \le\ & c\,2^{r_0 l}\sum_{m=0}^{r_0} \|E_{li}\tilde{f} - PE_{li}\tilde{f}\|_{C^m(U)} \ \le\ c\,2^{r_0 l}|E_{li}\tilde{f}|_{r_0,U}. \end{aligned} \tag{160}$$

Finally,

$$\begin{aligned}\max_{i\in\mathcal{I}_l} |E_{li}\tilde{f}|_{r_0,U} &= 2^{-r_0 l}\max_{i\in\mathcal{I}_l} |\tilde{f}|_{r_0,U_{li}} \le 2^{-r_0 l}|\tilde{f}|_{r_0,\mathbb{R}^{d_0}} \\ &\le 2^{-r_0 l}\|\tilde{f}\|_{C^{r_0}(\mathbb{R}^{d_0})} \le c\,2^{-r_0 l}\|f\|_{C^{r_0}(Q)}. \end{aligned} \tag{161}$$

Combining (157) and (160–161), we obtain

$$\|T_l f\|_{C^{r_0}(Q)} \le \|f\|_{C^{r_0}(Q)} + \|f - T_l f\|_{C^{r_0}(Q)} \le c\|f\|_{C^{r_0}(Q)},$$

which concludes the proof of (148) for $Z = \mathbb{R}$.

Now let Z be an arbitrary Banach space and let T_l be defined by (144) for Z, while $T_l^{\mathbb{R}}$ denotes the respective operator for $\mathbb{R}$. Using the already shown scalar case, the general case of (147) follows analogously to (139). The Banach space case of (148) is derived as

$$\begin{aligned}
\|T_l f\|_{C^{r_0}(Q,Z)} &= \max_{0\le j\le r_0} \sup_{z^*\in B_{Z^*}} \left\| \left(\frac{d^j(T_l f)}{ds^j}, z^* \right) \right\|_{C(Q,\mathscr{L}_j(\mathbb{R}^{d_0},\mathbb{R}))} \\
&= \sup_{z^*\in B_{Z^*}} \max_{0\le j\le r_0} \left\| \frac{d^j\left(T_l^{\mathbb{R}}(f,z^*)\right)}{ds^j} \right\|_{C(Q,\mathscr{L}_j(\mathbb{R}^{d_0},\mathbb{R}))} \\
&= \sup_{z^*\in B_{Z^*}} \left\| T_l^{\mathbb{R}}(f,z^*) \right\|_{C^{r_0}(Q)} \\
&\le c_2 \sup_{z^*\in B_{Z^*}} \|(f,z^*)\|_{C^{r_0}(Q)} = c_2\|f\|_{C^{r_0}(Q,Z)},
\end{aligned}$$

where for the latter relation we refer to the last part of (140).

□

We also need the following result.

Lemma 4.6. *There are constants $c_1, c_2 > 0$ such that for all $1 \le p \le 2$, $p \le q < \infty$, for all $n \in \mathbb{N}$, and for any Banach space Z and measure space (M,μ) the following hold:*

$$\tau_p(L_q(M,\mu,Z)) \le c_1\sqrt{q}\,\tau_p(Z) \tag{162}$$

$$\tau_p(\ell_\infty^n(Z)) \le c_2\sqrt{\log(n+1)}\,\tau_p(Z). \tag{163}$$

Proof. We start with (162). Let $(g_i)_{i=1}^m \subset L_q(M,\mu,Z)$. Then we have

$$\begin{aligned}
\left(\mathbb{E}\left\| \sum_{i=1}^m \varepsilon_i g_i \right\|_{L_q(M,\mu,Z)}^p \right)^{q/p} &\le \mathbb{E}\left\| \sum_{i=1}^m \varepsilon_i g_i \right\|_{L_q(M,\mu,Z)}^q \\
&= \int_M \mathbb{E}\left\| \sum_{i=1}^m \varepsilon_i g_i(t) \right\|_Z^q d\mu(t), \qquad (164)
\end{aligned}$$

with $(\varepsilon_i)_{i=0}^m$ a sequence of independent centered Bernoulli random variables. Next we apply the equivalence of moments and the type inequality to obtain

$$\int_M \mathbb{E}\Big\| \sum_{i=1}^m \varepsilon_i g_i(t)\Big\|_Z^q d\mu(t) \tag{165}$$

$$\leq (c\sqrt{q})^q \int_M \Big(\mathbb{E}\Big\| \sum_{i=1}^m \varepsilon_i g_i(t)\Big\|_Z^p\Big)^{q/p} d\mu(t) \tag{166}$$

$$\leq (c\sqrt{q}\,\tau_p(Z))^q \int_M \Big(\sum_{i=1}^m \|g_i(t)\|_Z^p\Big)^{q/p} d\mu(t) \tag{167}$$

with a constant $c > 0$ independent of p and q (see, e.g., [23], p. 100, for the step from (165) to (166)). Using the triangle inequality in $L_{q/p}(M,\mu)$, we get

$$\int_M \Big(\sum_{i=1}^m \|g_i(t)\|_Z^p\Big)^{q/p} d\mu(t)$$

$$\leq \left(\sum_{i=1}^m \left(\int_M \|g_i(t)\|_Z^q d\mu(t)\right)^{p/q}\right)^{q/p} = \left(\sum_{i=1}^m \|g_i\|_{L_q(M,\mu,Z)}^p\right)^{q/p}. \tag{168}$$

Joining (164), (167), and (168) yields (162). To show (163) we note that the identity map $I_Z : \ell_q^n(Z) \to \ell_\infty^n(Z)$ satisfies

$$\|I_Z\| = 1, \quad \|I_Z^{-1}\| = n^{1/q}. \tag{169}$$

If $n \geq 4$, we set $q = \log n$, so $q \geq 2 \geq p$ and $n^{1/q} \leq 2$. For $n < 4$ we put $q = 2$. Now (163) follows from (162) and (169).

□

Proof of Theorem 4.1. Our goal is to apply Proposition 2.3 with $X = C(Q,Z)$ and $Y = C^{r_0}(Q,Z)$. Using that $\Gamma_k \subseteq \Gamma_l$ for $k \leq l$, it follows from (122) and (149) of Lemma 4.5 that

$$P_k T_l = P_k \quad (k \leq l). \tag{170}$$

We put for $l \in \mathbb{N}_0$

$$X_l = T_l(C(Q,Z)) \subset C(Q,Z), \quad Y_l = T_l(C^{r_0}(Q,Z)) \subset C^{r_0}(Q,Z),$$

so $X_l = Y_l$ algebraically, but X_l is endowed with the norm induced by $C(Q,Z)$ and Y_l with the norm induced by $C^{r_0}(Q,Z)$. Next we derive estimates of $\tau_p(X_l)$ and $\tau_p(Y_l)$. For $i \in \mathcal{I}_l$ we let V_{li} be the linear vector space

$$V_{li} = \operatorname{span}\left\{\eta_{lk} R_{lk}\varphi_j|_{Q_{li}} \,:\, k \in \mathcal{I}_l \text{ with } U_{lk}^0 \cap Q_{li} \neq \emptyset,\ j = 1,\dots,\kappa\right\}.$$

Furthermore, let

$$X_{li} = \left(V_{li}, \|\,\|_{C(Q_{li})}\right), \qquad Y_{li} = \left(V_{li}, \|\,\|_{C^{r_0}(Q_{li})}\right),$$

$$\tilde{X}_{li} = \left(V_{li} \otimes Z, \|\,\|_{C(Q_{li},Z)}\right), \quad \tilde{Y}_{li} = \left(V_{li} \otimes Z, \|\,\|_{C^{r_0}(Q_{li},Z)}\right),$$

where $\otimes$ denotes the algebraic tensor product. We observe that by (145), for $f \in C(Q,Z)$

$$T_l f|_{Q_{li}} \in \tilde{X}_{li}.$$

Moreover, for $m = 0, r_0$

$$\|T_l f\|_{C^m(Q,Z)} = \max_{i\in\mathcal{I}_l} \left\|T_l f|_{Q_{li}}\right\|_{C^m(Q_{li},Z)}.$$

Consequently, X_l can be identified isometrically with a subspace of

$$\tilde{X}_l = \left(\bigoplus_{i\in\mathcal{I}_l} \tilde{X}_{li}\right)_\infty \tag{171}$$

and Y_l with a subspace of

$$\tilde{Y}_l = \left(\bigoplus_{i\in\mathcal{I}_l} \tilde{Y}_{li}\right)_\infty . \tag{172}$$

It follows from (141) and (151) that there is a constant $c > 0$ such that for all $l \in \mathbb{N}_0$, $i \in \mathcal{I}_l$

$$d_{li} := \dim V_{li} \le c. \tag{173}$$

Two Banach spaces Z_1 and Z_2 are called c-isomorphic, where $c \ge 1$, if there is an isomorphism $T : Z_1 \to Z_2$ with $\|T\|\|T^{-1}\| \le c$. The Banach-Mazur distance $d(Z_1, Z_2)$ between Z_1 and Z_2 is defined to be the infimum of all such c. Next we show that there is a constant $c > 0$ such that

$$d(\tilde{X}_{li}, \ell_\infty^{d_{li}}(Z)) \le c, \quad d(\tilde{Y}_{li}, \ell_\infty^{d_{li}}(Z)) \le c \quad (l \in \mathbb{N}_0, i \in \mathcal{I}_l). \tag{174}$$

Indeed, it suffices to consider $\tilde{Y}_{li}$, the case $\tilde{X}_{li}$ follows by setting $r_0 = 0$. Let $(g_k)_{k=1}^{d_{li}}$ be an Auerbach basis of Y_{li}, that is,

$$\max_{1\le k\le d_{li}} |\alpha_k| \le \left\|\sum_{k=1}^{d_{li}} \alpha_k g_k\right\|_{Y_{li}} \le \sum_{k=1}^{d_{li}} |\alpha_k| \quad (\alpha_k \in \mathbb{R},\ k = 1, \dots, d_{li}). \tag{175}$$

Such bases exist in every finite dimensional Banach space, see [24], Prop. 1.c.3. Now define $T : V_{li} \otimes Z \to \ell_\infty^{d_{li}}(Z)$ for

$$w = \sum_{k=1}^{d_{li}} g_k \otimes z_k \in V_{li} \otimes Z$$

by $Tw = (z_k)_{k=1}^{d_{li}}$. Then

$$\begin{aligned}
\|w\|_{\tilde{Y}_{li}} &= \left\|\sum_{k=1}^{d_{li}} g_k \otimes z_k\right\|_{\tilde{Y}_{li}} = \max_{0\le j\le r_0} \max_{z^*\in B_{Z^*}} \left\|\sum_{k=1}^{d_{li}} (z_k, z^*) \frac{d^j g_k}{ds^j}\right\|_{C(Q_{li},\mathscr{L}_j(\mathbb{R}^{d_0},\mathbb{R}))} \\
&= \max_{z^*\in B_{Z^*}} \left\|\sum_{k=1}^{d_{li}} (z_k, z^*)\, g_k\right\|_{C^{r_0}(Q_{li})} = \max_{z^*\in B_{Z^*}} \left\|\sum_{k=1}^{d_{li}} (z_k, z^*)\, g_k\right\|_{Y_{li}} .
\end{aligned}$$

Moreover, using (175), it follows that

$$\begin{aligned}
\|Tw\|_{\ell_\infty^{d_{li}}(Z)} &= \max_{1\le k\le d_{li}} \|z_k\| = \max_{z^*\in B_{Z^*}} \max_{1\le k\le d_{li}} |\,(z_k, z^*)\,| \\
&\le \max_{z^*\in B_{Z^*}} \left\| \sum_{k=1}^{d_{li}} (z_k, z^*)\, g_k \right\|_{Y_{li}} \\
&\le \max_{z^*\in B_{Z^*}} \sum_{k=1}^{d_{li}} |\,(z_k, z^*)\,| \le d_{li} \max_{1\le k\le d_{li}} \|z_k\| = d_{li}\|Tw\|_{\ell_\infty^{d_{li}}(Z)},
\end{aligned}$$

hence $\|T\|\|T^{-1}\| \le d_{li}$, which together with (173) gives the second relation of (174).

From (173) we get

$$m_l := \sum_{i\in\mathcal{I}_l} d_{li} \le c2^{d_0 l} \quad (l \in \mathbb{N}_0). \tag{176}$$

It follows from (171), (172), and (174) that

$$d(\tilde{X}_l, \ell_\infty^{m_l}(Z)) \le c, \quad d(\tilde{Y}_l, \ell_\infty^{m_l}(Z)) \le c \quad (l \in \mathbb{N}_0),$$

and therefore

$$\begin{aligned}
\tau_p(X_l) &\le \tau_p(\tilde{X}_l) \le c\tau_p(\ell_\infty^{m_l}(Z)) \\
\tau_p(Y_l) &\le \tau_p(\tilde{Y}_l) \le c\tau_p(\ell_\infty^{m_l}(Z)).
\end{aligned}$$

This together with Lemma 4.6 and (176) implies that there is a constant $c > 0$ such that for all $l \in \mathbb{N}_0$

$$\tau_p(Y_l) \le c(l+1)^{1/2}\tau_p(Z), \quad \tau_p(X_l) \le c(l+1)^{1/2}\tau_p(Z). \tag{177}$$

Furthermore, if $f \in \mathscr{C}_{\mathrm{Lip}}^{0,0,0}(Q \times [a,b] \times Z, Z; \kappa, L)$, we get from (146) and (149) that for all $l \in \mathbb{N}_0$, $t \in [a,b]$, $x \in C(Q,Z)$

$$\begin{aligned}
T_l\bar{f}(t,x) &= \sum_{s\in\Gamma_l} \big(\bar{f}(t,x)\big)(s)\,\zeta_{ls} = \sum_{s\in\Gamma_l} f(s,t,x(s))\,\zeta_{ls} \\
&= \sum_{s\in\Gamma_l} f(s,t,(T_l x)(s))\,\zeta_{ls} = \sum_{s\in\Gamma_l} \big(\bar{f}(t,T_l x)\big)(s)\,\zeta_{ls} \\
&= T_l\bar{f}(t,T_l x).
\end{aligned} \tag{178}$$

Similarly, for all $x \in C(Q,Z)$ and $s \in Q$

$$(\bar{f}(t,x), \delta_s) = f(s,t,x(s)) = f(s,t,(x,\delta_s)) = f_s(t,(x,\delta_s)).$$

By Lemma 3.2, $\bar f \in \mathcal{C}^{0,0}_{\rm Lip}([a,b]\times C(Q,Z), C(Q,Z);\kappa_1, L_1)$ for some κ_1, L_1 and by definition, see (48–50), $f_s \in \mathcal{C}^{0,0}_{\rm Lip}([a,b]\times Z, Z;\kappa, L)$. Now we apply Lemma 2.2 with $T=\delta_s$ and $g=f_s$ and obtain

$$\begin{aligned} \left(A^r_{n_{l_0},\omega}(\bar f, u_0), \delta_s\right) &= A^r_{n_{l_0},\omega}(f_s, u_0(s)) \quad (s\in Q) &(179)\\ (A^{r_1}_{n_l,\omega}(\bar f, u_0), \delta_s) &= A^{r_1}_{n_l,\omega}(f_s, u_0(s)) \quad (s\in Q,\, l_0 < l \le l_1). &(180)\end{aligned}$$

As a consequence of (25), (123), (179), and (180), we can relate algorithm $\mathscr{A}_\omega$ for the parametric problem to algorithm A_ω for the general Banach space valued problem of Section 2 as follows

$$\begin{aligned} \mathscr{A}_\omega(f,u_0) &= P_{l_0}\left(\left(A^r_{n_{l_0},\omega}(f_s,u_0(s))\right)_{s\in\Gamma_{l_0}}\right)\\ &\quad + \sum_{l=l_0+1}^{l_1}(P_l - P_{l-1})\left(\left(A^{r_1}_{n_l,\omega}(f_s,u_0(s))\right)_{s\in\Gamma_l}\right)\\ &= P_{l_0}A^r_{n_{l_0},\omega}(\bar f,u_0) + \sum_{l=l_0+1}^{l_1}(P_l-P_{l-1})A^{r_1}_{n_l,\omega}(\bar f,u_0)\\ &= A_\omega(\bar f,u_0). &(181)\end{aligned}$$

We put

$$\mathcal{K}_0 = \{(\bar f, u_0):\, (f,u_0)\in\mathscr{F}\}.$$

Then Proposition 3.1 gives

$$\begin{aligned}\mathcal{K}_0 &\subseteq \mathcal{F}^{r,\varrho}([a,b]\times C(Q,Z), C(Q,Z);\kappa_1, L_1, \sigma, \lambda_1)\\ &\quad \cap\, \mathcal{F}^{r_1,\varrho_1}([a,b]\times C^{r_0}(Q,Z), C^{r_0}(Q,Z);\kappa_1, L_1,\sigma,\lambda_1). &(182)\end{aligned}$$

Furthermore, (137), (147–148), (170), (178), and (182) show that the assumptions of Proposition 2.3 are fulfilled. Therefore (31) of Proposition 2.3 together with (61), (138), and (181) prove (125). The estimate (126) follows from (32) of Proposition 2.3 together with (61), (138), (177), and (181).

□

5 Complexity

We work in the setting of information-based complexity theory, as discussed in [29, 27]. For details on the notions used here we refer to [13, 14]. An abstract numerical problem is described by a tuple $\mathcal{P} = (F, G, S, K, \Lambda)$. The set F is the set of input data, in our case $F=\mathscr{F}$, G is a normed linear space and $S: F\to G$ an (in general nonlinear) operator, the solution operator, which maps the input $\psi\in F$ to the exact solution $S(\psi)$. In our case we have $G = B(Q\times[a,b], Z)$ and

$S = \mathscr{S}$. Furthermore, K is a nonempty set and Λ a set of mappings from F to K, the set of information functionals. In our case K is Z and Λ is given by

$$\Lambda = \{\delta_{s,t,z} : \ s \in Q, t \in [a,b], z \in Z\} \cup \{\delta_s : \ s \in Q\}, \tag{183}$$

where for $(f, u_0) \in \mathscr{F}$

$$\delta_{s,t,z}(f, u_0) = f(s,t,z), \quad \delta_s(f, u_0) = u_0(s). \tag{184}$$

So the admissible information is Z-valued and consists of values of f and u_0.

Below $e_n^{\rm det}(\mathscr{S}, \mathscr{F})$ and $e_n^{\rm ran}(\mathscr{S}, \mathscr{F})$ denote the n-th minimal error of $\mathscr{S}$ on $\mathscr{F}$ in the deterministic, respectively randomized setting, that is, the minimal possible error among all deterministic, respectively randomized algorithms, that use at most n information functionals.

The following theorem, which is the main result of this paper, gives almost sharp estimates of the deterministic and randomized minimal errors and hence, of the complexity of the parametric initial value problem. Moreover, combined with Corollary 4.3, it shows that the upper bounds are realized by the multilevel algorithm presented before, more precisely, in the deterministic case by $\mathscr{A}_\omega$ for any $\omega \in \Omega$, and in the randomized case, by $(\mathscr{A}_\omega)_{\omega\in\Omega}$, with parameters chosen in an appropriate way. Concerning the assumption $r + \varrho \ge r_1 + \varrho_1$, we refer to Remark 4.2.

Theorem 5.1. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $0 \le \varrho, \varrho_1 \le 1$, with $r + \varrho \ge r_1 + \varrho_1$, $\kappa, L : (0, +\infty) \to (0, +\infty)$, and $\sigma, \lambda > 0$, where we assume that*

$$\kappa_0 := \inf_{0<R<+\infty} \kappa(R) > 0. \tag{185}$$

Let Z be a Banach space, and let $\mathscr{F}$ be defined by (57–59). Then in the deterministic setting,

$$n^{-\upsilon_1} \preceq e_n^{\rm det}(\mathscr{S}, \mathscr{F}) \preceq_{\log} n^{-\upsilon_1}, \tag{186}$$

where υ_1 was defined in (131).

Moreover, let $1 \le p \le 2$ and assume that Z is of type p. Let p_Z denote the supremum of all p_1 such that Z is of type p_1. Then in the randomized setting,

$$n^{-\upsilon_2(p_Z)} \preceq e_n^{\rm ran}(\mathscr{S}, \mathscr{F}) \preceq_{\log} n^{-\upsilon_2(p)}, \tag{187}$$

with υ_2 given by (133).

It is readily seen from (133) that $\upsilon_2(p)$ is a continuous, monotonically increasing function of $p \in [1, 2]$. It follows that the bounds in the randomized case of Theorem 5.1 are matching up to an arbitrarily small gap in the exponent. Under additional assumptions, upper and lower bounds are of the same order up to logarithmic factors.

Corollary 5.2. *Assume that the conditions of Theorem 5.1 hold. Let* p_Z *be the supremum of all* p_1 *such that* Z *is of type* p_1. *Then for each* $\varepsilon > 0$

$$n^{-\upsilon_2(p_Z)} \preceq e_n^{\rm ran}(\mathscr{S}, \mathscr{F}) \preceq n^{-\upsilon_2(p_Z)+\varepsilon}.$$

If, moreover, the supremum of types is attained, that is, Z *is of type* p_Z, *then*

$$n^{-\upsilon_2(p_Z)} \preceq e_n^{\rm ran}(\mathscr{S}, \mathscr{F}) \preceq_{\log} n^{-\upsilon_2(p_Z)}.$$

The latter assumption is satisfied, in particular, by spaces of type 2 and, if $1 \le p_1 < \infty$, *by spaces* $X = L_{p_1}(M, \mu)$, *where* (M, μ) *is some measure space.*

Proof of Theorem 5.1. The upper bounds follow from Corollary 4.3. To show the lower bounds, let $\mathscr{S}_0 : C(Q \times [a, b], Z) \to B(Q, Z)$ be given for $f \in C(Q \times [a, b], Z)$ by

$$(\mathscr{S}_0 f)(s) = \int_a^b f(s, t)dt \quad (s \in Q).$$

This is the operator of Z-valued definite parametric integration, with a one-dimensional integration domain. Define

$$V_1 : C(Q \times [a, b], Z) \to C(Q \times [a, b], Z) \times C(Q, Z)$$

for $f \in C(Q \times [a, b], Z)$ by

$$V_1 f = (f, 0) \tag{188}$$

and

$$V_2 : B(Q \times [a, b], Z) \to B(Q, Z)$$

for $g \in B(Q \times [a, b], Z)$ by

$$(V_2 g)(s) = g(s, b) \quad (s \in Q).$$

Then we have

$$\|V_2\| = 1. \tag{189}$$

For $f \in C(Q \times [a, b], Z)$ (considering functions on $Q \times [a, b]$ as functions on $Q \times [a, b] \times Z$ not depending on $z \in Z$) the solution $u = \mathscr{S}(f, 0)$ of

$$\begin{aligned} \frac{d}{dt} u(s, t) &= f(s, t) \quad (s \in Q, t \in [a, b]) \\ u(s, a) &= 0 \quad (s \in Q) \end{aligned}$$

is

$$u(s, t) = \int_a^t f(s, \tau)d\tau.$$

Consequently,

$$\mathscr{S}_0 = V_2 \circ \mathscr{S} \circ V_1. \tag{190}$$

Moreover, if f satisfies

$$\|f\|_{C(Q\times[a,b],Z)} \le (b-a)^{-1}\lambda, \tag{191}$$

then

$$\sup_{s\in Q, t\in[a,b]} \|u(s,t)\| \le \lambda. \tag{192}$$

Furthermore, according to (12–14), for $n\in\mathbb{N}$, $\omega\in\Omega$, $s\in Q$ we have $A^0_{n,\omega}(f_s,0) = v(s,\,\cdot\,)$ with

$$v(s,t) = \begin{cases} p_{k,0}(s,t) & \text{if} \quad t\in[t_k,t_{k+1}),\ 0\le k\le n-1, \\ u_n(s) & \text{if} \quad t=t_n, \end{cases}$$

$u_0(s)=0$, and for $0\le k\le n-1$, $t\in[t_k,t_{k+1}]$

$$\begin{aligned} p_{k,0}(s,t) &= u_k(s) + (t-t_k)f(s,t_k) \\ u_{k+1}(s) &= u_k(s) + hf(s,\xi_{k+1}). \end{aligned}$$

So (191) also implies

$$\sup_{s\in Q} \left\|A^0_{n,\omega}(f_s,0)\right\|_{B([a,b],Z)} \le \lambda. \tag{193}$$

Let φ_0 be a C^∞ function on $\mathbb{R}^{d_0}$ with support in Q and $\sup_{s\in Q}|\varphi_0(s)| = \sigma_0 > 0$, and let $m_0\in\mathbb{N}$. We divide the cube Q into $m_0^{d_0}$ congruent subcubes Q_i ($i=1,\dots,m_0^{d_0}$) of disjoint interior. Let s_i be the point in Q_i with minimal Euclidean norm and define

$$\varphi_{0,i}(s) = \varphi_0(m_0(s-s_i)) \quad (s\in Q,\ i=1,\dots,m_0^{d_0}).$$

Furthermore, let φ_1 be a C^∞ function on $\mathbb{R}$ with support in $[a,b]$ and $\left|\int_a^b \varphi_1(t)dt\right| = \sigma_1 > 0$. For $m_1\in\mathbb{N}$ we let $t_j = a + j(b-a)/m_1$ and

$$\varphi_{1,j}(t) = \varphi_1(a + m_1(t-t_j)) \quad (t\in[a,b],\ j=0,\dots,m_1-1).$$

Finally, let $(z_j)_{j=0}^{m_1-1}\subset B_Z$ be any sequence (to be specified later on) and define

$$\psi_{ij}(s,t) = \varphi_{0,i}(s)\varphi_{1,j}(t)z_j.$$

Denote $\mathscr{I}_{m_0,m_1} = \{1,\dots,m_0^{d_0}\}\times\{0,\dots,m_1-1\}$ and set

$$\Psi^0_{m_0,m_1} = \left\{ \sum_{(i,j)\in\mathscr{I}_{m_0,m_1}} \delta_{ij}\psi_{ij} :\ \delta_{ij}\in[-1,1],\ (i,j)\in\mathscr{I}_{m_0,m_1} \right\}. \tag{194}$$

Taking into account (185), we observe that there is a constant $c_0 > 0$ such that for all $m_0, m_1 \in \mathbb{N}$,

$$
\begin{aligned}
c_0 m_1^{-r-\varrho}\Psi^0_{m_0,m_1} &\subseteq \mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q\times[a,b]\times Z, Z;\kappa, L) && (195)\\
c_0 m_0^{-r_0} m_1^{-r_1-\varrho_1}\Psi^0_{m_0,m_1} &\subseteq \mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q\times[a,b]\times Z, Z;\kappa, L) && (196)\\
c_0\Psi^0_{m_0,m_1} &\subseteq (b-a)^{-1}\lambda B_{C(Q\times[a,b],Z)}. && (197)
\end{aligned}
$$

We put

$$
\Psi_{m_0,m_1} = c_0 \min\left(m_1^{-r-\varrho}, m_0^{-r_0}m_1^{-r_1-\varrho_1}\right)\Psi^0_{m_0,m_1}, \qquad (198)
$$

thus, by (195–197)

$$
\begin{aligned}
\Psi_{m_0,m_1} &\subseteq \mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q\times[a,b]\times Z, Z;\kappa, L)\\
&\qquad \cap\, \mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q\times[a,b]\times Z, Z;\kappa, L) && (199)\\
\Psi_{m_0,m_1} &\subseteq (b-a)^{-1}\lambda B_{C(Q\times[a,b],Z)}. && (200)
\end{aligned}
$$

Using (199–200) and (191–193), it follows that for all $m_0, m_1 \in \mathbb{N}$,

$$
V_1\left(\Psi_{m_0,m_1}\right) \subseteq \mathscr{F}. \qquad (201)
$$

We put $K_0 = Z$ and consider the following class of information functionals on $C(Q\times[a,b], Z)$

$$
\Lambda_0 = \{\delta_{s,t} : s\in Q, t\in[a,b]\}, \quad \delta_{s,t}(f) = f(s,t). \qquad (202)
$$

We conclude from (190) and (201) that the problem

$$
(\mathscr{S}_0, \Psi_{m_0,m_1}, B(Q,Z), Z, \Lambda_0)
$$

reduces to

$$
(\mathscr{S}, \mathscr{F}, B(Q\times[a,b],Z), Z, \Lambda)
$$

(see Section 3 of [14]). Consequently, by (189), for all $n, m_0, m_1 \in \mathbb{N}$

$$
e_n^{\mathrm{set}}(\mathscr{S},\mathscr{F}) \ \geq\ e_n^{\mathrm{set}}(\mathscr{S}_0, \Psi_{m_0,m_1}), \qquad (203)
$$

where set $\in \{\mathrm{det}, \mathrm{ran}\}$. Moreover, by linearity and (198),

$$
e_n^{\mathrm{set}}(\mathscr{S}_0, \Psi_{m_0,m_1}) = c_0 \min\left(m_1^{-r-\varrho}, m_0^{-r_0}m_1^{-r_1-\varrho_1}\right) e_n^{\mathrm{set}}(\mathscr{S}_0, \Psi^0_{m_0,m_1}). \qquad (204)
$$

For all $\delta_{ij} \in \mathbb{R}$ $((i,j)\in\mathscr{I}_{m_0,m_1})$ we have

$$
\begin{aligned}
\left\| \mathscr{S}_0 \sum_{(i,j)\in\mathscr{I}_{m_0,m_1}} \delta_{ij}\psi_{ij}\right\|_{B(Q,Z)} &= \left\| \sum_{(i,j)\in\mathscr{I}_{m_0,m_1}} \delta_{ij}\varphi_{0,i} z_j \int_a^b \varphi_{1,j}(t)dt\right\|_{B(Q,Z)}\\
&= \sigma_0\sigma_1 m_1^{-1} \max_{1\le i\le m_0^{d_0}} \left\| \sum_{j=0}^{m_1-1} \delta_{ij} z_j\right\|. && (205)
\end{aligned}
$$

Now we prove the lower bounds in the deterministic setting. Here we take any $w_0 \in Z$ with $\|w_0\| = 1$ and set $z_j = w_0$ $(j = 0, \ldots, m_1 - 1)$. Using standard results, see [29], Ch. 4.5, as well as (205), we obtain

$$\begin{aligned} & e_n^{\rm det}(\mathscr{S}_0, \Psi^0_{m_0,m_1}) \\ & \ge \min_{\mathscr{I} \subseteq \mathscr{I}_{m_0,m_1},\, |\mathscr{I}| \ge m_0^{d_0} m_1 - n} \Big\| \mathscr{S}_0 \sum_{(i,j)\in\mathscr{I}} \psi_{ij} \Big\| \\ & \ge \sigma_0 \sigma_1 m_1^{-1} \min_{\mathscr{I} \subseteq \mathscr{I}_{m_0,m_1},\, |\mathscr{I}| \ge m_0^{d_0} m_1 - n} \max_{1 \le i \le m_0^{d_0}} |\{j : (i,j) \in \mathscr{I}\}| \ge c, \quad (206) \end{aligned}$$

provided $m_0^{d_0} m_1 \ge 2n$. If $r_0/d_0 > r_1 + \varrho_1$, we set

$$m_0 = \left\lceil 2n^{\frac{r+\varrho-r_1-\varrho_1}{r_0+(r+\varrho-r_1-\varrho_1)d_0}} \right\rceil, \quad m_1 = \left\lceil n^{\frac{r_0}{r_0+(r+\varrho-r_1-\varrho_1)d_0}} \right\rceil$$

and get from (203–204), (206), and (131)

$$\begin{aligned} e_n^{\rm det}(\mathscr{S}, \mathscr{F}) & \ge c \min\left(m_1^{-r-\varrho}, m_0^{-r_0} m_1^{-r_1-\varrho_1}\right) \\ & \ge c n^{-\frac{r_0(r+\varrho)}{r_0+(r+\varrho-r_1-\varrho_1)d_0}} = c n^{-\upsilon_1}. \end{aligned}$$

If $r_0/d_0 \le r_1 + \varrho_1$, we put $m_0 = \lceil 2n^{1/d_0} \rceil$, $m_1 = 1$, and derive similarly

$$e_n^{\rm det}(\mathscr{S}, \mathscr{F}) \ \ge \ c n^{-\frac{r_0}{d_0}} = c n^{-\upsilon_1},$$

which proves the lower bound in (186).

Finally we consider the randomized setting. Lemma 5 and 6 of [13] with $K = Z$ (Lemma 6 is formulated for $K = \mathbb{R}$, but is easily seen to hold also for $K = Z$) give

$$e_n^{\rm ran}(\mathscr{S}_0, \Psi^0_{m_0,m_1}) \ \ge \ \frac{1}{4} \min_{\mathscr{I} \subseteq \mathscr{I}_{m_0,m_1},\, |\mathscr{I}| \ge m_0^{d_0} m_1 - 4n} \mathbb{E} \Big\| \mathscr{S}_0 \sum_{(i,j)\in\mathscr{I}} \varepsilon_{ij} \psi_{ij} \Big\|$$

with $\{\varepsilon_{ij} : (i,j) \in \mathscr{I}_{m_0,m_1}\}$ being independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_{ij} = -1\} = \mathbb{P}\{\varepsilon_{ij} = +1\} = 1/2$. Using (205), we conclude for $m_0^{d_0} m_1 \ge 8n$

$$\begin{aligned} & e_n^{\rm ran}(\mathscr{S}_0, \Psi^0_{m_0,m_1}) \\ & \ge \frac{\sigma_0\sigma_1}{4m_1} \min_{\mathscr{I} \subseteq \mathscr{I}_{m_0,m_1},\, |\mathscr{I}| \ge m_0^{d_0} m_1 - 4n} \mathbb{E} \max_{1 \le i \le m_0^{d_0}} \Big\| \sum_{j:\,(i,j)\in\mathscr{I}} \varepsilon_{ij} z_j \Big\| \\ & \ge \frac{\sigma_0\sigma_1}{4m_1} \min_{\mathscr{I} \subseteq \mathscr{I}_{m_0,m_1},\, |\mathscr{I}| \ge m_0^{d_0} m_1 - 4n} \max_{1 \le i \le m_0^{d_0}} \mathbb{E} \Big\| \sum_{j:\,(i,j)\in\mathscr{I}} \varepsilon_{ij} z_j \Big\|. \quad (207) \end{aligned}$$

Now we distinguish between two cases. If $p_Z = 2$, we use the same choice $z_j = w_0$ as in the deterministic setting. Then by Khintchine's inequality, see [24], Th. 2.b.3,

$$\mathbb{E} \Big\| \sum_{j:\,(i,j)\in\mathscr{I}} \varepsilon_{ij} z_j \Big\| \ge c \, |\{j : (i,j) \in \mathscr{I}\}|^{1/2}. \quad (208)$$

If $p_Z < 2$, Z must be infinite dimensional, because a finite dimensional space Z always satisfies $p_Z = 2$. By the Maurey-Pisier Theorem (see [26], Th. 2.3), there is a sequence $(w_j)_{j=0}^{m_1-1} \subset Z$ such that for all $(\delta_j)_{j=0}^{m_1-1} \subset \mathbb{R}$

$$\frac{1}{2}\left(\sum_{j=0}^{m_1-1}|\delta_j|^{p_Z}\right)^{1/p_Z} \le \left\|\sum_{j=0}^{m_1-1}\delta_j w_j\right\| \le \left(\sum_{j=0}^{m_1-1}|\delta_j|^{p_Z}\right)^{1/p_Z}.$$

Setting $z_j = w_j$ $(j = 0, \dots, m_1 - 1)$, we get

$$\mathbb{E}\left\|\sum_{j:\,(i,j)\in\mathscr{I}}\varepsilon_{ij}z_j\right\| \ge \frac{1}{2}\left|\{j:\,(i,j)\in\mathscr{I}\}\right|^{1/p_Z}. \tag{209}$$

Assuming $m_0^{d_0}m_1 \ge 8n$, we obtain from (207–209) for both cases

$$\begin{aligned} & e_n^{\mathrm{ran}}(\mathscr{S}_0, \Psi^0_{m_0,m_1}) \\ \ge\ & c m_1^{-1} \min_{\mathscr{I}\subseteq\mathscr{I}_{m_0,m_1},\,|\mathscr{I}|\ge m_0^{d_0}m_1-4n}\ \max_{1\le i\le m_0^{d_0}} \left|\{j:\,(i,j)\in\mathscr{I}\}\right|^{1/p_Z} \\ \ge\ & c m_1^{-1+1/p_Z}. \end{aligned} \tag{210}$$

Combining (203), (204), and (210), it follows that for $m_0^{d_0}m_1 \ge 8n$

$$e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \ge c m_1^{-1+1/p_Z}\min\left(m_1^{-r-\varrho}, m_0^{-r_0}m_1^{-r_1-\varrho_1}\right). \tag{211}$$

If $r_0/d_0 > r_1 + \varrho_1 + 1 - 1/p_Z$, we define

$$m_0 = \left\lceil 8n^{\frac{r+\varrho-r_1-\varrho_1}{r_0+(r+\varrho-r_1-\varrho_1)d_0}}\right\rceil, \quad m_1 = \left\lceil n^{\frac{r_0}{r_0+(r+\varrho-r_1-\varrho_1)d_0}}\right\rceil,$$

which together with (211) and (133) gives

$$e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \ \ge\ c n^{-\frac{r_0(r+\varrho+1-1/p_Z)}{r_0+(r+\varrho-r_1-\varrho_1)d_0}} = c n^{-\upsilon_2(p_Z)}.$$

If $r_0/d_0 \le r_1 + \varrho_1 + 1 - 1/p_Z$, we set $m_0 = \lceil 8n^{1/d_0}\rceil$, $m_1 = 1$, and get from (211) and (133)

$$e_n^{\mathrm{ran}}(\mathscr{S},\mathscr{F}) \ \ge\ c n^{-\frac{r_0}{d_0}} = c n^{-\upsilon_2(p_Z)},$$

which shows the lower bound in (187).

$\square$

6 Special classes of functions

First let us consider the case of globally bounded functions. Here we have $\kappa \equiv \kappa_0$ and $L \equiv L_0$ with $\kappa_0, L_0 \in \mathbb{R}$, $\kappa_0, L_0 > 0$. Then

$$\begin{aligned} \mathscr{F} &= \left(\mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q\times[a,b]\times Z, Z; \kappa_0, L_0)\cap\mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q\times[a,b]\times Z, Z; \kappa_0, L_0)\right) \\ &\quad \times \sigma B_{C^{r_0}(Q,Z)}, \end{aligned}$$

provided the constant λ involved in the definition (57–59) of $\mathscr{F}$ satisfies

$$\lambda \geq \sigma + \kappa_0(b-a). \tag{212}$$

In other words, for globally bounded classes conditions (58) and (59) are automatically fulfilled whenever (212) holds.

Next let us consider the case of linear equations and see how it fits the class $\mathscr{F}$. For $\kappa_0 > 0$ let $C^{r_0,r,\varrho}(Q \times [a,b], Z; \kappa_0)$ denote the subset of all functions in $\mathscr{C}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z; \kappa_0)$ which do not depend on $z \in Z$. Given $\kappa_0, \kappa_1, \sigma > 0$, let $\mathscr{G}$ be the set of all pairs (f, u_0) with $u_0 \in \sigma B_{C^{r_0}(Q,Z)}$ and $f : Q \times [a,b] \times Z \to Z$ of the form

$$f(s,t,z) = g_0(s,t) + g_1(s,t)z \tag{213}$$

with

$$\begin{aligned} g_0 &\in C^{0,r,\varrho}(Q \times [a,b], Z; \kappa_0) \cap C^{r_0,r_1,\varrho_1}(Q \times [a,b], Z; \kappa_0) && (214)\\ g_1 &\in C^{0,r,\varrho}(Q \times [a,b], \mathscr{L}(Z); \kappa_1) \cap C^{r_0,r_1,\varrho_1}(Q \times [a,b], \mathscr{L}(Z); \kappa_1). && (215) \end{aligned}$$

This means we consider the linear equation

$$\begin{aligned} \frac{d}{dt}u(s,t) &= g_0(s,t) + g_1(s,t)u(s,t) && (216)\\ u(s,a) &= u_0(s). && (217) \end{aligned}$$

Corollary 6.1. *Let $r_0, r, r_1 \in \mathbb{N}_0$, $0 \leq \varrho, \varrho_1 \leq 1$, with $r+\varrho \geq r_1+\varrho_1$, $\kappa_0, \kappa_1, \sigma > 0$. Then there exist $\kappa, L : (0,+\infty) \to (0,+\infty)$ and $\lambda > 0$ such that*

$$\mathscr{G} \subseteq \mathscr{F} \tag{218}$$

where $\mathscr{G}$ is defined in (213–215) *and $\mathscr{F}$ in* (57–59), *and the statements of Theorem* 5.1 *hold with $\mathscr{F}$ replaced by $\mathscr{G}$.*

Proof. It is easily checked that

$$\begin{aligned} \mathscr{G} \subseteq \Big(&\mathscr{C}^{0,r,\varrho}_{\mathrm{Lip}}(Q \times [a,b] \times Z, Z; \kappa, L) \\ &\cap \mathscr{C}^{r_0,r_1,\varrho_1}_{\mathrm{Lip}}(Q \times [a,b] \times Z, Z; \kappa, L)\Big) \times \sigma B_{C^{r_0}(Q,Z)} \end{aligned}$$

for suitable κ, L. Thus, it remains to verify (58–59). Since f is Lipschitz with constant κ_1, the solution of (216–217) exists on $[a,b]$ and is unique. Integrating with respect to t we get

$$u(s,t) = u_0(s) + \int_a^t (g_0(s,\tau) + g_1(s,\tau)u(s,\tau))\, d\tau,$$

consequently, for $t \in [a,b]$

$$\|u(\,\cdot\,,t)\|_{B(Q,Z)} \leq \sigma + (b-a)\kappa_0 + \kappa_1 \int_a^t \|u(\,\cdot\,,\tau)\|_{B(Q,Z)}\, d\tau,$$

which, by Gronwall's lemma, gives

$$\|u\|_{B(Q\times[a,b],Z)} \le (\sigma + (b-a)\kappa_0)e^{\kappa_1(b-a)}.$$

By (12–14) we have $A^0_{n,\omega}(f_s, u_0(s)) = v(s,\,\cdot\,)$, where

$$v(s,t) = \begin{cases} p_{k,0}(s,t) & \text{if} \quad t\in[t_k,t_{k+1}) \quad \text{and} \quad 0\le k\le n-1, \\ u_n(s) & \text{if} \quad t = t_n, \end{cases}$$

and for $0 \le k \le n-1$, $t\in[t_k,t_{k+1}]$

$$p_{k,0}(s,t) = u_k(s) + (t-t_k)g_0(s,t_k) + (t-t_k)g_1(s,t_k)u_k(s) \tag{219}$$

$$u_{k+1}(s) = u_k(s) + hg_0\big(s,\xi_{k+1}\big) + hg_1\big(s,\xi_{k+1}\big)p_{k,0}(s,\xi_{k+1}). \tag{220}$$

Inserting (219) with $t = \xi_{k+1}$ into (220), we get

$$\begin{aligned} u_{k+1}(s) &= u_k(s) + hg_0\big(s,\xi_{k+1}\big) + h(\xi_{k+1}-t_k)g_1\big(s,\xi_{k+1}\big)g_0(s,t_k) \\ &\quad + hg_1\big(s,\xi_{k+1}\big)u_k(s) + h(\xi_{k+1}-t_k)g_1\big(s,\xi_{k+1}\big)g_1(s,t_k)u_k(s), \end{aligned}$$

thus with $c_0 = \kappa_0(1+h\kappa_1)$, $c_1 = \kappa_1(1+h\kappa_1)$

$$\|u_{k+1}\|_{B(Q,Z)} \le (1+c_1h)\,\|u_k\|_{B(Q,Z)} + c_0h.$$

Using $\|u_0\|_{B(Q,Z)} \le \sigma$, we obtain for $1\le k\le n$

$$\begin{aligned} \|u_k\|_{B(Q,Z)} &\le \sigma(1+c_1h)^k + c_0h\sum_{j=0}^{k-1}(1+c_1h)^j \\ &\le (\sigma + c_0/c_1)(1+c_1h)^n \le (\sigma+\kappa_0/\kappa_1)e^{c_1(b-a)} \\ &\le (\sigma+\kappa_0/\kappa_1)\,e^{\kappa_1(1+(b-a)\kappa_1)(b-a)}. \end{aligned}$$

Together with (219) this implies

$$\begin{aligned} &\max_{0\le k\le n-1}\ \max_{t\in[t_k,t_{k+1}]} \|p_{k,0}(\,\cdot\,,t)\|_{B(Q,Z)} \\ &\quad\le (1+(b-a)\kappa_1)(\sigma+\kappa_0/\kappa_1)e^{\kappa_1(1+(b-a)\kappa_1)(b-a)} + (b-a)\kappa_0 \end{aligned}$$

and hence the desired result (218), which, in turn, implies the upper bound.

That the lower bounds of Theorem 5.1 also hold for $\mathscr{G}$ follows directly from the proof of Theorem 5.1 and the fact that $\mathscr{G}$ contains all pairs $(f,0)$ with

$$f = g_0 \in C^{0,r,\varrho}(Q\times[a,b],Z;\kappa_0)\cap C^{r_0,r_1,\varrho_1}(Q\times[a,b],Z;\kappa_0).$$

□

Now let us motivate the choice of the smoothness for the class $\mathscr{F}$ in (57–59). This is best explained when looking at the subset of those functions f which depend only on s and t. Then the parameters $r_0, r, r_1, \varrho, \varrho_1$ describe the smoothness of $f(s,t)$ and we arrive for $Z = \mathbb{R}$ at classes analogous to those studied in [6] (so we also refer to the discussion in Section 5 of that paper).

The smoothness we imposed with respect to z can be considered as chosen in a 'complementary' way. By this we mean the following. As we showed in Section 5, the complexity only mildly depends on the smoothness in z in the sense that increasing this smoothness does not result in a higher rate of the minimal errors. In fact, even if f does not depend on z at all, we get the same rate. Therefore, with the smoothness parameters $r_0, r, r_1, \varrho, \varrho_1$ set for s and t, the smoothness in z has been chosen in such a way that it just guarantees the respective convergence rate. (Of course, a challenging problem is to find minimal smoothness requirements in z still ensuring the same rate. We do not pursue this aspect here.)

The class $\mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z; \kappa, L)$ consists of functions with a certain type of dominating mixed smoothness. We have chosen $\mathscr{F}$ to be given by an intersection of two such classes, because this way we can also include isotropic smoothness and certain anisotropic analogues thereof. Let us look at these special cases in some more detail. For the subsequent discussion we assume, for the sake of simplicity, that Z is of type 2, which includes, in particular, the case of finite systems of scalar equations $Z = \mathbb{R}^d$.

First we consider the case $r = r_1$, $\varrho = \varrho_1$. Then $\mathscr{F}$ is the set of all

$$(f, u_0) \in \mathscr{C}_{\mathrm{Lip}}^{r_0,r,\varrho}(Q \times [a,b] \times Z, Z; \kappa, L) \times \sigma B_{C^{r_0}(Q,Z)}$$

satisfying (58) and, if $r = \varrho = 0$, (59). Thus, the involved functions f have dominating mixed smoothness. From Theorem 5.1 we obtain

Corollary 6.2. *Let $r_0, r \in \mathbb{N}_0$, $0 \le \varrho \le 1$, $r = r_1$, $\varrho = \varrho_1$, assume that (185) holds and that Z is of type 2. Then*

$$e_n^{\mathrm{det}}(\mathscr{S}, \mathscr{F}) \asymp_{\log} n^{-\min\left(r+\varrho,\, \frac{r_0}{d_0}\right)}$$

$$e_n^{\mathrm{ran}}(\mathscr{S}, \mathscr{F}) \asymp_{\log} n^{-\min\left(r+\varrho+\frac{1}{2},\, \frac{r_0}{d_0}\right)}.$$

Hence, if $r_0/d_0 \le r+\varrho$, the rates are the same. If $r_0/d_0 > r+\varrho$, the best rate of randomized algorithms is superior to that of deterministic algorithms. If $r_0/d_0 \ge r + \varrho + \frac{1}{2}$, the best randomized algorithms outperform the best deterministic ones by an order of $n^{-1/2}$. This is particularly important if, e.g., $r = 0$ and ϱ is small. Then the deterministic rate $n^{-\varrho}$ is slow (for $\varrho = 0$, there is no convergence rate at all), while in the randomized setting we still have at least $n^{-1/2}$.

Next we assume $r_1 = \varrho_1 = 0$, which means that $\mathscr{F}$ is the set of

$$(f, u_0) \in \Big(\mathscr{C}_{\mathrm{Lip}}^{0,r,\varrho}(Q \times [a,b] \times Z, Z; \kappa, L) \cap \mathscr{C}_{\mathrm{Lip}}^{r_0,0,0}(Q \times [a,b] \times Z, Z; \kappa, L)\Big) \times \sigma B_{C^{r_0}(Q,Z)} \qquad (221)$$

fulfilling (58) and, if $r = \varrho = 0$, (59), so that here the functions f have smoothness in s and t separately. In this case Theorem 5.1 yields

Corollary 6.3. *Let $r_0, r \in \mathbb{N}_0$, $0 \le \varrho \le 1$, $r_1 = \varrho_1 = 0$, suppose (185) holds and Z is of type 2. Then*

$$\begin{aligned} e_n^{\mathrm{det}}(\mathscr{S}, \mathscr{F}) &\asymp_{\log} n^{-\upsilon_1} \\ e_n^{\mathrm{ran}}(\mathscr{S}, \mathscr{F}) &\asymp_{\log} n^{-\upsilon_2}, \end{aligned}$$

where

$$\upsilon_1 = \begin{cases} 0 & \text{if } \quad r_0 = 0 \\ \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+r+\varrho}\,(r+\varrho) & \text{otherwise} \end{cases} \tag{222}$$

$$\upsilon_2 = \begin{cases} \frac{\frac{r_0}{d_0}}{\frac{r_0}{d_0}+r+\varrho}\left(r+\varrho+\frac{1}{2}\right) & \text{if } \quad \frac{r_0}{d_0} \ge \frac{1}{2} \\ \frac{r_0}{d_0} & \text{if } \quad \frac{r_0}{d_0} < \frac{1}{2}. \end{cases} \tag{223}$$

Except for the trivial case $r_0 = 0$, the randomized setting is always superior to the deterministic one, although the maximum of improvement $n^{-1/2}$ is only reached if $r = \varrho = 0$ and $r_0/d_0 \ge 1/2$ (this case, in fact, has already been considered above).

Next we keep the restriction $r_1 = \varrho_1 = 0$ and assume also $\varrho = 0$. In this case we want to identify certain subclasses of $\mathscr{F}$. Let $r_2 \in \mathbb{N}_0$ and let $\mathscr{C}^{[r_0,r,r_2]}(Q \times [a,b] \times Z, Z; \kappa)$ be the space of continuous functions $f : Q \times [a,b] \times Z \to Z$ having for all $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}_0^3$ with

$$\frac{\alpha_0}{r_0} + \frac{\alpha_1}{r} + \frac{\alpha_2}{r_2} \le 1 \tag{224}$$

(we interpret $\frac{0}{0} = 0$ and $\frac{\tau}{0} = +\infty$ if $\tau > 0$) continuous partial derivatives $\frac{\partial^{|\alpha|} f(s,t,z)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}}$ such that for $R > 0$, $s \in Q$, $t \in [a,b]$, $z \in RB_Z$

$$\left\| \frac{\partial^{|\alpha|} f(s,t,z)}{\partial s^{\alpha_0} \partial t^{\alpha_1} \partial z^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0}, Z, Z)} \le \kappa(R).$$

If $r_0 = r_2 = r$, then this is just isotropic C^r-smoothness. Furthermore, if $r_2 \ge r_0$, we let $\mathscr{C}_{\mathrm{Lip}}^{[r_0,r,r_2]}(Q \times [a,b] \times Z, Z; \kappa, L)$ be the subset consisting of those $f \in \mathscr{C}_{\mathrm{Lip}}^{[r_0,r,r_2]}(Q \times [a,b] \times Z, Z; \kappa)$ which satisfy the Lipschitz conditions

$$\left\| \frac{\partial^{|\alpha|} f(s,t,z_1)}{\partial s^{\alpha_0} \partial z^{\alpha_2}} - \frac{\partial^{|\alpha|} f(s,t,z_2)}{\partial s^{\alpha_0} \partial z^{\alpha_2}} \right\|_{\mathscr{L}_{\alpha_0,\alpha_2}(\mathbb{R}^{d_0}, Z, Z)} \le L(R) \|z_1 - z_2\|$$

for $\alpha_0 + \alpha_2 \le r_0$, $R > 0$, $s \in Q$, $t \in [a,b]$, $z_1, z_2 \in RB_Z$. Finally, we let $\mathscr{H}$ be the set of all

$$(f, u_0) \in \mathscr{C}_{\mathrm{Lip}}^{[r_0,r,r_2]}(Q \times [a,b] \times Z, Z; \kappa, L) \times \sigma B_{C^r(Q,Z)}$$

satisfying (58) and, if $r = 0$, (59). Considering the class $\mathscr{F}$ with $r_1 = \varrho_1 = \varrho = 0$ and taking into account (221), it follows that for all $r_2 \ge \max(r_0, r)$

$$\mathscr{H} \subseteq \mathscr{F}. \tag{225}$$

Corollary 6.4. *Let $r_0, r, r_2 \in \mathbb{N}_0$, $r_2 \ge \max(r_0, r)$, assume that (185) holds and that Z is of type 2. Then for any $\mathscr{M}$ with $\mathscr{H} \subseteq \mathscr{M} \subseteq \mathscr{F}$*

$$\begin{aligned} e_n^{\det}(\mathscr{S}, \mathscr{M}) &\asymp_{\log} n^{-\upsilon_1} \\ e_n^{\mathrm{ran}}(\mathscr{S}, \mathscr{M}) &\asymp_{\log} n^{-\upsilon_2}, \end{aligned}$$

where υ_1 and υ_2 are as in (222–223), with $\varrho = 0$. In particular, if $r_0 = r > 0$, υ_1 and υ_2 take the form

$$\upsilon_1 = \frac{r}{d_0 + 1}, \quad \upsilon_2 = \begin{cases} \frac{r+\frac{1}{2}}{d_0+1} & \text{if } \frac{r}{d_0} \ge \frac{1}{2} \\ \frac{r}{d_0} & \text{if } \frac{r}{d_0} < \frac{1}{2}. \end{cases} \tag{226}$$

Proof. The upper bounds follow from (225) and Corollary 6.3. Let us show the lower bounds. We use the notation from the proof of Theorem 5.1. There is a constant $c_0 > 0$ such that for all $m_0, m_1 \in \mathbb{N}$, $\psi \in \Psi^0_{m_0,m_1}$

$$\begin{aligned} &\|\psi\|_{C^{[r_0,r,r_2]}(Q\times[a,b]\times Z, Z)} \\ \le\ & c_0 \max\left\{ m_0^{\alpha_0} m_1^{\alpha_1} : \alpha_0, \alpha_1 \in \mathbb{N}_0,\ \frac{\alpha_0}{r_0} + \frac{\alpha_1}{r} \le 1 \right\} \\ =\ & c_0 \max\left\{ \left(m_0^{r_0}\right)^{\frac{\alpha_0}{r_0}} \left(m_1^{r}\right)^{\frac{\alpha_1}{r}} : \alpha_0, \alpha_1 \in \mathbb{N}_0,\ \frac{\alpha_0}{r_0} + \frac{\alpha_1}{r} \le 1 \right\} \\ \le\ & c_0 \max\left(m_0^{r_0}, m_1^{r}\right). \end{aligned}$$

Setting

$$\Psi_{m_0,m_1} = \min\left(\kappa_0, (b-a)^{-1}\lambda\right) c_0^{-1} \min(m_0^{-r_0}, m_1^{-r})\, \Psi^0_{m_0,m_1},$$

it follows that

$$\begin{aligned} \Psi_{m_0,m_1} &\subseteq \mathscr{C}^{[r_0,r,r_2]}_{\mathrm{Lip}}(Q \times [a,b] \times Z, Z; \kappa, L) \\ \Psi_{m_0,m_1} &\subseteq (b-a)^{-1}\lambda B_{C(Q\times[a,b],Z)} \end{aligned}$$

and therefore, by (192) and (193), for all $m_0, m_1 \in \mathbb{N}$

$$V_1\left(\Psi_{m_0,m_1}\right) \subseteq \mathscr{H} \subseteq \mathscr{M},$$

where V_1 is defined in (188). Now the same argument as used in the proof of Theorem 5.1 (with $r_1 = \varrho_1 = \varrho = 0$) gives the lower bounds.

□

As we already discussed above in regard to $\mathscr{F}$, also here the rate does not depend on the smoothness r_2 of f in the variable z. We observe that by (226), for $r_0 = r$ and in particular in the isotropic case $r_0 = r_2 = r$, the maximal speedup of randomized algorithms over deterministic ones is $n^{-1/4}$, reached for $d_0 = 1$, $r \geq 1$.

Finite systems of d scalar ODEs are included in our analysis by setting $Z = \ell_2^d$. Letting $\mathscr{F}_\infty$ stand for $\mathscr{F}$ with $Z = \ell_2(\mathbb{N})$ and denoting the classes $\mathscr{F}$ for $Z = \ell_2^d$ by $\mathscr{F}_d$ (all with the same dimension of the parameter space d_0 and with the same constants $\kappa, L, \sigma, \lambda$), it is easily shown that $\mathscr{F}_d$ can be embedded into $\mathscr{F}_\infty$ in a uniform way. This shows, in particular, that the error estimates of the algorithm, see Corollary 4.3, hold with constants which are independent of the dimension d of the system. Taking into account that an ℓ_2^d-valued information functional is equivalent to d scalar-valued information functionals, it follows that the family $(\mathscr{F}_d)_{d\in\mathbb{N}}$ is polynomially tractable in the randomized setting if $r_0 > 0$ and in the deterministic setting if $r_0 > 0$ and $r + \varrho > 0$. We refer to [28] for the notion of tractability and more on this direction of research.

In the present paper we did not strive for the best estimates in terms of the involved powers of logarithmic factors, as already made clear in Remark 4.4, since for general Banach spaces even the exponent is only known up to an arbitrary small $\varepsilon > 0$. We also left out the question of ODEs defined in bounded domains, since in general Banach spaces standard localization methods do not work due to the non-existence of smooth bump functions, see [9]. Both topics – bounded domains and sharp asymptotic estimates – will be covered in a subsequent paper [7] for the case of Z being a Hilbert space, including this way finite scalar systems.

References

[1] H. Cartan, Cours de calcul différentiel, Hermann, Paris, 1977.

[2] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[3] K. Deimling, Ordinary Differential Equations in Banach Spaces, Lecture Notes in Mathematics 596, Springer, Berlin, 1977.

[4] Th. Daun, On the randomized solution of initial value problems, J. Complexity 27 (2011), 300–311.

[5] Th. Daun, S. Heinrich, Complexity of Banach space valued and parametric integration, to appear in the proceedings of Monte Carlo and Quasi-Monte Carlo Methods 2012.

[6] Th. Daun, S. Heinrich, Complexity of parametric integration in various smoothness classes, submitted.

[7] Th. Daun, S. Heinrich, Complexity of parametric initial value problems for systems of ODEs, in preparation.

[8] H. O. Fattorini, Second Order Linear Differential Equations in Banach Spaces, North Holland Mathematical Studies 108, Amsterdam, 1985.

[9] G. Godefroy, Renorming of Banach spaces, in: W. B. Johnson, J. Lindenstrauss, eds., Handbook of the Geometry of Banach Spaces, Vol.1, Elsevier, Amsterdam 2001, pp. 781–835 .

[10] M. Hansen, C. Schwab, Sparse adaptive approximation of high dimensional parametric initial value problems, Vietnam J. Math. 41 (2013), 181–215.

[11] S. Heinrich, Monte Carlo complexity of global solution of integral equations, J. Complexity 14 (1998), 151–175.

[12] S. Heinrich, The multilevel method of dependent tests, Advances in Stochastic Simulation Methods (N. Balakrishnan, V. B. Melas, S. M. Ermakov, editors), pp. 47–62, Birkhäuser, Boston, Basel, Berlin 2000.

[13] S. Heinrich, Monte Carlo approximation of weakly singular integral operators, J. Complexity 22 (2006), 192–219.

[14] S. Heinrich, The randomized information complexity of elliptic PDE, J. Complexity 22 (2006), 220–249.

[15] S. Heinrich, Complexity of initial value problems in Banach spaces, J. Math. Phys. Anal. Geom. 9 (2013), 73–101.

[16] S. Heinrich, B. Milla, The randomized complexity of initial value problems, J. Complexity 24 (2008), 77–88.

[17] S. Heinrich, E. Sindambiwe, Monte Carlo complexity of parametric integration, J. Complexity 15 (1999), 317–341.

[18] L. Hörmander, The Analysis of Linear Partial Differential Operators, I, Springer, Berlin, 1983.

[19] B. Kacewicz, How to increase the order to get minimal-error algorithms for systems of ODE, Numer. Math. 45 (1984), 93–104.

[20] B. Kacewicz, Randomized and quantum algorithms yield a speed-up for initial-value problems, J. Complexity 20 (2004), 821–834, see also http://arXiv.org/abs/quant-ph/0311148.

[21] B. Kacewicz, Almost optimal solution of initial-value problems by randomized and quantum algorithms, J. Complexity 22 (2006), 676–690, see also http://arXiv.org/abs/quant-ph/0510045.

[22] S. G. Krein, Linear Differential Equations in Banach Spaces, Amer. Math. Soc., Providence, 1971.

[23] M. Ledoux, M. Talagrand, Probability in Banach Spaces, Springer, Berlin, 1991.

[24] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces, 1: Sequence Spaces, Springer, Berlin, 1977.

[25] R. H. Martin, Nonlinear Operators and Differential Equations in Banach Spaces, John Wiley, New York, 1976.

[26] B. Maurey, G. Pisier, Series de variables aléatoires vectorielles independantes et propriétés geométriques des espaces de Banach, Stud. Math. 58 (1976), 45–90.

[27] E. Novak, Deterministic and Stochastic Error Bounds in Numerical Analysis, Lecture Notes in Mathematics 1349, Springer, Berlin, 1988.

[28] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volumes 1-3, European Math. Soc., Zürich, 2008, 2010, 2012.

[29] J. F. Traub, G. W. Wasilkowski, and H. Woźniakowski, Information-Based Complexity, Academic Press, New York, 1988.

[30] H. Whitney, Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc. 36 (1934), 63–89.

[31] C. Wiegand, Optimal Monte Carlo and Quantum Algorithms for Parametric Integration, PhD Thesis 2005, University of Kaiserslautern; Shaker Verlag Aachen, 2006.

[32] O. A. Zautykov, K. G. Valeev, Infinite Systems of Differential Equations, Nauka, Alma-Ata, 1974.