Monte Carlo Approximation of Weakly Singular Integral Operators

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Abstract

We study the randomized approximation of weakly singular integral operators. For a suitable class of kernels having a standard type of singularity and being otherwise of finite smoothness, we develop a Monte Carlo multilevel method, give convergence estimates and prove lower bounds which show the optimality of this method and establish the complexity. As an application we obtain optimal methods for and the complexity of randomized solution of the Poisson equation in simple domains, when the solution is sought on subdomains of arbitrary dimension.

1 Introduction

In a number of papers Monte Carlo methods for the computation of integrals depending on a parameter, integral operators and the solution of integral equations were proposed and studied, see [3, 18, 14, 15, 16, 20, 21]. The complexity of these problems in the randomized setting was investigated in [5, 6, 10]. There a new type of Monte Carlo methods – multilevel variance reduction – was introduced and shown to be optimal for such problems. These multilevel methods assumed the smoothness of the integrand (kernel) in the whole domain, while typical kernels in applications often possess (weak) singularities.

In the present paper we study this situation. We propose a multilevel Monte Carlo method for the approximation of integral operators, which takes care of the singularity. We analyze its convergence rate, prove lower bounds, determine the complexity of the problem and establish optimality of the method.

As an application we study the following model problem: the solution of the Poisson equation in a d-dimensional ball, with homogeneous Dirichlet boundary conditions, and the solution being sought on a subcube of arbitrary dimension. Optimal algorithms are derived.

Basic facts on Monte Carlo methods can be found in [2, 11, 14, 15]. For general background on the theory of information-based complexity, within the frame of which we carry out our investigations, we refer to [17, 19, 4].

2 Preliminaries

We shall use the following notation. Let $d \in \mathbb{N}$ (where $\mathbb{N}$ always means $\{1, 2, \dots\}$, while $\mathbb{N}_0$ stands for $\mathbb{N} \cup \{0\}$). For a bounded Lebesgue measurable set $Q \subset \mathbb{R}^d$ of positive Lebesgue measure we let $L_\infty(Q)$ denote the space of essentially bounded real-valued Lebesgue measurable functions on Q, endowed with the essential supremum norm. If $Q \subset \mathbb{R}^d$ is closed and bounded, we let $C(Q)$ be the space of continuous functions on Q, equipped with the supremum norm. If, moreover, Q is the closure of its interior points, and $s \in \mathbb{N}$, we let $C^s(Q)$ be the space of continuous real functions on Q which are s-times continuously differentiable in the interior Q^0 of Q, and whose partial derivatives up to order s have continuous extensions to Q. The norm on $C^s(Q)$ is defined as

$$\|f\|_{C^s(Q)} = \max_{|\alpha| \le s} \sup_{x \in Q} |D^\alpha f(x)|.$$

The subspace of $C(Q)$ (respectively, of $C^s(Q)$) consisting of those functions which vanish (respectively, vanish together with all derivatives up to order s) on the boundary of Q is denoted by $C_0(Q)$ (respectively, $C_0^s(Q)$). For normed spaces X and Y we let $L(X, Y)$ denote the space of all bounded linear operators from X to Y, and $B_X = \{u \in X : \|u\|_X \le 1\}$ the unit ball.

Let us introduce the problem we study. Given two sets $M, Q \subset R^d$, a kernel function k on $M \times Q$ and a function $f \in L_\infty(Q)$, we seek to approximate

$$(T_k f)(x) = \int_Q k(x, y) f(y) dy \quad (x \in M),$$

considered as an operator into $L_\infty(M)$, that is, the error being measured in the norm of $L_\infty(M)$. Now let us specify the assumptions.

Let $d_1, d \in \mathbb{N}$, $d_1 \leq d$, let $M = [0,1]^{d_1}$ be the d_1-dimensional unit cube and let $Q \subset \mathbb{R}^d$ be bounded, Lebesgue measurable, and of positive Lebesgue measure. In the case $d_1 < d$ we shall identify M with the subset $[0,1]^{d_1} \times \{0^{(d-d_1)}\}$ of $\mathbb{R}^d$. In this sense, let $\operatorname{diag}(M,Q) := \{(x,x) : x \in M \cap Q\}$.

Next let us specify the class of kernels. The following notation will be helpful. For $\tau \in \mathbb{R}$ and $x \neq y \in \mathbb{R}^d$ define

$$\gamma_\tau(x,y) = \begin{cases} |x-y|^\tau & \text{if} \quad \tau < 0, \\ |\ln|x-y|| + 1 & \text{if} \quad \tau = 0, \\ 1 & \text{if} \quad \tau > 0. \end{cases} \tag{1}$$

Let $s \in \mathbb{N}$ and $\sigma \in \mathbb{R}$, $-d < \sigma < +\infty$. We introduce the following set of kernels $\mathcal{C}^{s,\sigma}(M,Q)$. It consists of all Lebesgue measurable functions $k : M \times Q \setminus \operatorname{diag}(M,Q) \to \mathbb{R}$ with the property that there is a constant $c > 0$ such that for all $y \in Q$

1. $k(x,y)$ is s-times continuously differentiable with respect to x on $M^0 \setminus \{y\}$, where M^0 means the interior of M, as a subset of $\mathbb{R}^{d_1}$,

2. for all multiindices $\alpha \in N_0^{d_1}$ with $0 \leq |\alpha| = \alpha_1 + \cdots + \alpha_{d_1} \leq s$ the α-th partial derivative of k with respect to the x-variables, which we denote by $D_x^\alpha k(x,y)$, satisfies the estimate

$$|D_x^\alpha k(x,y)| \leq c\, \gamma_{\sigma-|\alpha|}(x,y) \quad (x \in M^0 \setminus \{y\}), \tag{2}$$

   and

3. for all $\alpha \in N_0^{d_1}$ with $0 \leq |\alpha| \leq s$ the functions $D_x^\alpha k(x,y)$ have continuous extensions to $M \setminus \{y\}$.

For the sake of completeness we also want to include the case $d_1 = 0$ into some of the results. Here we put $M = \{0\} \subset R^d$. The set $\mathcal{C}^{s,\sigma}(M,Q)$ does not depend on s and consists of all functions $k(0,y)$ which are Lebesgue measurable in y and satisfy

$$|k(0,y)| \leq c\gamma_\sigma(0,y) \quad (y \in Q \setminus \{0\}) \tag{3}$$

for a certain $c > 0$. The target space $L_\infty(M)$ is then understood as replaced by $\mathbb{R}$, that is, the operator T_k acts from $L_\infty(Q)$ to $\mathbb{R}$.

For $k \in \mathcal{C}^{s,\sigma}(M,Q)$ let $\|k\|_{\mathcal{C}^{s,\sigma}}$ denote the smallest $c > 0$ satisfying (2). It is easily checked that $\|\,.\,\|_{\mathcal{C}^{s,\sigma}}$ is a norm, which turns $\mathcal{C}^{s,\sigma}(M,Q)$ into a

Banach space. Examples of kernels in $\mathcal{C}^{s,\sigma}(M,Q)$ include the weakly singular kernels

$$k(x,y) = h(x,y)|x-y|^{\sigma},$$

for $-d < \sigma < +\infty$, $\sigma \notin \{0, 2, 4, \dots\}$, and

$$k(x,y) = h(x,y)|x-y|^{\sigma} \ln |x-y|,$$

for even $\sigma \geq 0$, where h is Lebesgue measurable on $M \times Q$, $h(\,.\,,y)$ is in $C^s(M)$ for all $y \in Q$ and

$$\sup_{y \in Q} \|h(\,.\,,y)\|_{C^s(M)} < \infty.$$

This is easily checked by differentiation. In particular, for $m \in \mathbb{N}$, the fundamental solution of Δ^m, the m-th power of the Laplacian in $\mathbb{R}^d$, has, up to a constant factor, the form

$$|x-y|^{2m-d}.$$

if $2m < d$ or d is odd, and

$$|x-y|^{2m-d} \ln |x-y|$$

if $2m \geq d$ and d is even. Clearly, if $k \in \mathcal{C}^{s,\sigma}(M,Q)$, then

$$T_k \in L(L_\infty(Q), L_\infty(M)).$$

In fact, T_k maps $L_\infty(Q)$ into $C(M)$, but since our approximation will be piecewise continuous, we prefer to work in $L_\infty(M)$.

3 The algorithm and its analysis

Throughout this section we assume $d_1 \geq 1$. First we present some approximation tools needed later. We are concerned with partitions, meshes and interpolation operators on $M = [0,1]^{d_1}$ exclusively. For $l = 0, 1, \dots$ let

$$M = \bigcup_{i=1}^{n_l} M_{li} \tag{4}$$

be the partition of M into

$$n_l = 2^{d_1 l}$$

closed subcubes of sidelength 2^{-l} and mutually disjoint interior. Let Γ_l be the equidistant mesh on M with mesh-size $2^{-l}(\max(1, s-1))^{-1}$ and $\Gamma_{li} = \Gamma_l \cap M_{li}$. Let $P_{li} : \ell_\infty(\Gamma_{li}) \to E_{li}$ be the multivariate (tensor product) Lagrange interpolation on Γ_{li}, where E_{li} is the space of multivariate polynomials on M_{li} of degree at most $s-1$ in each variable (thus, we consider the maximum degree). It is convenient for our purposes to identify E_{li} with a subspace of $L_\infty(M)$ by continuing the functions as $\equiv 0$ outside of M_{li}.

For our algorithm we also need the interpolation pieces of level $l+1$, collected on M_{li}. Put

$$\hat{E}_{li} = \sum_{j:\, M_{l+1,j} \subseteq M_{li}} E_{l+1,j},$$

(the sum is meant as a sum of subspaces of $L_\infty(M)$),

$$\hat{\Gamma}_{li} = \bigcup_{j:\, M_{l+1,j} \subseteq M_{li}} \Gamma_{l+1,j},$$

and define $\hat{P}_{li} : \ell_\infty(\hat{\Gamma}_{li}) \to \hat{E}_{li}$ by

$$\hat{P}_{li} u = \sum_{j:\, M_{l+1,j} \subseteq M_{li}} P_{l+1,j}(u|_{\Gamma_{l+1,j}}).$$

So $\hat{P}_{li}$ is just composite Lagrange interpolation (with respect to the pieces $M_{l+1,j} \subseteq M_{li}$). Note also that since we are working in $L_\infty(M)$, functions being equal except for a set of Lebesgue measure zero are identified. Set

$$E_l = \sum_{i=1}^{n_l} E_{li}.$$

Observe that this sum of subspaces is direct. The space E_l is just the space of piecewise polynomials on M of maximum degree at most $s-1$ with respect to the partition $(M_{li})_{i=1}^{n_l}$, with no correlation at the interfaces. Note further that $E_{li} \subset \hat{E}_{li} \subset E_{l+1}$ and $E_l \subset E_{l+1}$. Define $P_l : \ell_\infty(\Gamma_l) \to E_l$ by setting for $u \in \ell_\infty(\Gamma_l)$

$$P_l u = \sum_{i=1}^{n_l} P_{li}(u|_{\Gamma_{li}}).$$

Thus P_l is the corresponding piecewise Lagrange interpolation operator. For $f \in C(M_{li})$ or $f \in C(M)$ we write $P_{li} f$ instead of $P_{li}(f|_{\Gamma_{li}})$, and similarly $\hat{P}_{li} f$ and $P_l f$. Then we have

$$\hat{P}_{li} f = \sum_{j:\, M_{l+1,j} \subseteq M_{li}} P_{l+1,j} f \qquad (f \in C(M_{li}))$$

$$P_l f = \sum_{i=1}^{n_l} P_{li} f \qquad (f \in C(M))$$

$$P_{l+1} f = \sum_{i=1}^{n_l} \hat{P}_{li} f \qquad (f \in C(M)).$$

We need the following well-known properties of the operators just defined (see, e.g., [1]): There are constants $c_1, c_2, c_3 > 0$ such that for all l and i,

$$\|P_{li} : \ell_\infty(\Gamma_{li}) \to L_\infty(M)\| \le c_1 \tag{5}$$

and for $f \in C^s(M_{li})$,

$$\|f - P_{li} f\|_{L_\infty(M_{li})} \le c_2\, 2^{-sl} \|f\|_{C^s(M_{li})}, \tag{6}$$

and hence also

$$\|(\hat{P}_{li} - P_{li}) f\|_{L_\infty(M_{li})} \le c_3\, 2^{-sl} \|f\|_{C^s(M_{li})}. \tag{7}$$

Unless explicitly stated otherwise, throughout this paper constants are either absolute or may depend only on the problem parameters d_1, d, s, σ, Q, but neither on the input functions k and f nor on the algorithm parameters m, n, l, i etc. Furthermore, we often use the same symbol $c, c_1, \dots$ for possibly different positive constants (also when they appear in a sequence of relations). Finally, log always means $\log_2$.

Now we are ready to describe the algorithm. Fix any final level $m \in \mathbb{N}$. We shall approximate

$$\begin{aligned} T_k f \approx P_m T_k f &= P_0 T_k f + \sum_{l=0}^{m-1} (P_{l+1} - P_l) T_k f \\ &= P_0 T_k f + \sum_{l=0}^{m-1} \sum_{i=1}^{n_l} (\hat{P}_{li} - P_{li}) T_k f. \end{aligned} \tag{8}$$

To approximate $P_0 T_k f$, we need approximations of

$$(T_k f)(x) = \int_Q k(x,y) f(y)\, dy$$

for $x \in \Gamma_0$. Define

$$\bar{\varrho} = \sup\{|x - y| \,:\, x \in M,\, y \in Q\}. \tag{9}$$

In the sequel, $B(x,\varrho)$ will always denote the closed d-dimensional ball of radius ϱ around $x \in \mathbb{R}^d$. We shall use importance sampling. For this purpose, define for $x \in \Gamma_0$ a probability density on $B(x,\bar{\varrho})$ by setting

$$p_x^{(0)}(y) = \gamma_\sigma(x,y)/a^{(0)} \qquad (y \in B(x,\bar{\varrho}))$$

(recall the definition of γ_σ in (1)), where

$$a^{(0)} = \int_{B(x,\bar{\varrho})} \gamma_\sigma(x,y)dy = \int_{B(0,\bar{\varrho})} \gamma_\sigma(0,y)dy.$$

It follows that for $x \in \Gamma_0$

$$\begin{aligned}
\int_Q k(x,y)f(y)\,dy &= \int_{B(x,\bar{\varrho})} k(x,y)f(y)\chi_Q(y)\,dy \\
&= \int_{B(x,\bar{\varrho})} a^{(0)}k(x,y)f(y)\chi_Q(y)\gamma_\sigma^{-1}(x,y)p_x^{(0)}(y)\,dy \\
&= \int_{B(x,\bar{\varrho})} g^{(0)}(x,y)p_x^{(0)}(y)\,dy. \qquad (10)
\end{aligned}$$

Here $g^{(0)}(x,y)$ is defined for $x \in \Gamma_0$ and $y \in B(x,\bar{\varrho})$ by

$$g^{(0)}(x,y) = \begin{cases} a^{(0)}k(x,y)f(y)\gamma_\sigma^{-1}(x,y) & \text{if } \; y \in Q \setminus \{x\} \\ 0 & \text{otherwise.} \end{cases} \qquad (11)$$

Let $N^{(0)} \in \mathbb{N}$, to be fixed later on, let

$$\xi_{xj}^{(0)} \quad (x \in \Gamma_0,\ j = 1,\dots,N^{(0)})$$

be independent random variables with density $p_x^{(0)}$, on some probability space (Ω,Σ,μ). Our approximation to $(T_k f)(x)$ will be

$$\int_Q k(x,y)f(y)\,dy \approx \varphi_x^{(0)},$$

where

$$\varphi_x^{(0)} = \frac{1}{N^{(0)}} \sum_{j=1}^{N^{(0)}} g^{(0)}(x,\xi_{xj}^{(0)}) \quad (x \in \Gamma_0). \qquad (12)$$

Now we construct approximations for the summands in (8) corresponding to the l-levels. For $l = 0,1,\dots$ and $i = 1,\dots,n_l$ let x_{li} be the center of M_{li}

and set

$$
\begin{aligned}
\varrho_l &= \sqrt{d_1}2^{-l-1} \quad \text{(the radius of the sets } M_{li}\text{)}, \\
a_l &= \int_{|y|\le 3\varrho_l} \gamma_\sigma(0,y)dy = \int_{|y|\le 3\varrho_l} \begin{cases} |y|^\sigma dy & \text{if} \quad \sigma<0 \\ (|\ln|y||+1)\,dy & \text{if} \quad \sigma=0 \\ dy & \text{if} \quad \sigma>0. \end{cases} \qquad (13) \\
b_l &= \int_{2\varrho_l<|y|\le\bar\varrho} \gamma_{\sigma-s}(0,y)dy \\
&= \int_{2\varrho_l<|y|\le\bar\varrho} \begin{cases} |y|^{\sigma-s} dy & \text{if} \quad \sigma-s<0 \\ (|\ln|y||+1)\,dy & \text{if} \quad \sigma-s=0 \\ dy & \text{if} \quad \sigma-s>0. \end{cases} \qquad (14)
\end{aligned}
$$

Fix $l \in \{0,\dots,m-1\}$. We shall approximate

$$
(\hat P_{li} - P_{li})T_k f
$$

by constructing approximations of $(T_k f)(x)$ for $x \in \hat\Gamma_{li}$. We split the integral into a local, weakly singular part and a global, smooth part,

$$
(T_k f)(x) = \int_{B(x_{li},2\varrho_l)\cap Q} k(x,y)f(y)\,dy + \int_{Q\setminus B(x_{li},2\varrho_l)} k(x,y)f(y)\,dy, \qquad (15)
$$

each integral of which will be approximated separately by a Monte Carlo scheme, for $x \in \hat\Gamma_{li}$, using importance sampling again. For the first one, define for each $x \in \Gamma_{l+1}$

$$
p_{lx}(y) = a_l^{-1}\gamma_\sigma(x,y) \qquad (y \in B(x,3\varrho_l)), \qquad (16)
$$

which is a probability density on $B(x,3\varrho_l)$, since by (13),

$$
\int_{B(x,3\varrho_l)} \gamma_\sigma(x,y)dy = \int_{B(0,3\varrho_l)} \gamma_\sigma(0,y)dy = a_l.
$$

Observe that $B(x_{li},2\varrho_l) \subset B(x,3\varrho_l)$ for all $x \in \hat\Gamma_{li}$. We have

$$
\begin{aligned}
&\int_{B(x_{li},2\varrho_l)\cap Q} k(x,y)f(y)\,dy = \int_{B(x,3\varrho_l)} k(x,y)f(y)\chi_{B(x_{li},2\varrho_l)\cap Q}(y)\,dy \\
&= \int_{B(x,3\varrho_l)} a_l k(x,y)f(y)\chi_{B(x_{li},2\varrho_l)\cap Q}(y)\gamma_\sigma^{-1}(x,y)p_{lx}(y)\,dy \\
&= \int_{B(x,3\varrho_l)} g_{li}(x,y)p_{lx}(y)\,dy, \qquad (17)
\end{aligned}
$$

where $g_{li}(x,y)$ is defined for $x \in \hat{\Gamma}_{li}$ and $y \in B(x, 3\varrho_l)$ by

$$g_{li}(x,y) = \begin{cases} a_l k(x,y) f(y) \gamma_\sigma^{-1}(x,y) & \text{if} \quad y \in B(x_{li}, 2\varrho_l) \cap Q \setminus \{x\} \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Let $N_l \in \mathbb{N}$ $(l = 0, \ldots, m-1)$, also to be fixed later on, let

$$\xi_{lxj} \quad (l = 0, \ldots, m-1,\ x \in \Gamma_{l+1},\ j = 1, \ldots, N_l)$$

be independent (also of $\xi_{xj}^{(0)}$) random variables on (Ω, Σ, μ) with density p_{lx} given by (16). Our approximation to the first integral in (15) will be

$$\int_{B(x_{li}, 2\varrho_l) \cap Q} k(x,y) f(y)\, dy \approx \varphi_{lix},$$

with

$$\varphi_{lix} = \frac{1}{N_l} \sum_{j=1}^{N_l} g_{li}(x, \xi_{lxj}) \quad (x \in \hat{\Gamma}_{li}). \tag{19}$$

To approximate the second integral in (15) for $x \in \hat{\Gamma}_{li}$, we let

$$C_{li} = \{y \in \mathbb{R}^d :\ 2\varrho_l < |x_{li} - y| \le \bar{\varrho}\}.$$

where $\bar{\varrho}$ was defined in (9). Note that $Q \setminus B(x_{li}, 2\varrho_l) \subset C_{li}$. Thus, if $2\varrho_l \ge \bar{\varrho}$, the second integral in (15) is zero. If $2\varrho_l < \bar{\varrho}$, define a probability density q_{li} on C_{li} by setting

$$q_{li}(y) = b_l^{-1} \gamma_{\sigma-s}(x_{li}, y)$$

which is justified since

$$\int_{C_{li}} \gamma_{\sigma-s}(x_{li}, y) dy = \int_{2\varrho_l < |y| \le \bar{\varrho}} \gamma_{\sigma-s}(0, y) dy = b_l.$$

For any $x \in M_{li}$ we have

$$\begin{aligned} \int_{Q \setminus B(x_{li}, 2\varrho_l)} k(x,y) f(y)\, dy &= \int_{C_{li}} b_l k(x,y) f(y) \chi_Q(y) \gamma_{\sigma-s}^{-1}(x_{li}, y) q_{li}(y)\, dy \\ &= \int_{C_{li}} h_{li}(x,y) q_{li}(y)\, dy, \end{aligned} \tag{20}$$

where $h_{li}(x,y)$ is defined for $x \in M_{li}$ and $y \in C_{li}$ by

$$h_{li}(x,y) = \begin{cases} b_l k(x,y) f(y) \gamma_{\sigma-s}^{-1}(x_{li}, y) & \text{if} \quad y \in C_{li} \cap Q \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Let η_{lij} $(l = 0, \dots, m-1, i = 1, \dots, n_l, j = 1, \dots, N_l)$ be independent (also of $\xi_{xj}^{(0)}$ and ξ_{lij}) random variables with density q_{li}. We approximate for $x \in \hat{\Gamma}_{li}$

$$\int_{Q \setminus B(x_{li}, 2\varrho_l)} k(x,y) f(y)\, dy \approx \psi_{lix},$$

where

$$\psi_{lix} := \begin{cases} \frac{1}{N_l} \sum_{j=1}^{N_l} h_{li}(x, \eta_{lij}) & \text{if} \quad 2\varrho_l < \bar{\varrho} \\ 0 & \text{if} \quad 2\varrho_l \geq \bar{\varrho}. \end{cases} \tag{22}$$

Our final approximation will be

$$\theta = P_0 \left([\varphi_x^{(0)}]_{x \in \Gamma_0} \right) + \sum_{l=0}^{m-1} \sum_{i=1}^{n_l} (\hat{P}_{li} - P_{li}) \left([\varphi_{lix} + \psi_{lix}]_{x \in \hat{\Gamma}_{li}} \right). \tag{23}$$

This completes the description of the algorithm.

Now we analyze its error. We shall consider the expected mean square error

$$e(\theta) = (\mathbb{E} \, \| T_k f - \theta \|_{L_\infty(M)}^2)^{1/2}.$$

The cost of the algorithm θ is defined as

$$\operatorname{cost}(\theta) = N^{(0)} + \sum_{l=0}^{m-1} n_l N_l$$

— up to a constant this is the total number of needed function values (of k and f), arithmetic real number operations and random variables (of type ξ and η).

We need the following lemma, which is a consequence of Propostion 9.11 of Ledoux and Talagrand [13], see also [5].

Lemma 1. *There is a constant $c > 0$ such that if $n, N \in \mathbb{N}$ and $(\zeta_j)_{j=1}^N$ is a sequence of independent ℓ_∞^n–valued random variables with $\mathbb{E} \, \|\zeta_j\|_{\ell_\infty^n}^2 < \infty$ for all j, then*

$$\operatorname{Var}\Big(\sum_{j=1}^N \zeta_j \Big)_{\ell_\infty^n} \leq c \, \log n \sum_{j=1}^N \operatorname{Var}(\zeta_j)_{\ell_\infty^n}, \tag{24}$$

where $\operatorname{Var}(\zeta)_Z := \mathbb{E} \, \|\zeta - \mathbb{E} \, \zeta\|_Z^2$ denotes the variance of a random variable ζ with values in a Banach space Z.

To state the following proposition, define β (this parameter will describe the powers of the logarithmic term) as

$$\beta = \begin{cases} 0 & \text{if} \quad \min(s, d+\sigma) > \frac{d_1}{2} \\ \frac{3}{2} & \text{if} \quad \min(s, d+\sigma) = \frac{d_1}{2} \quad \text{and} \quad s \neq d+\sigma \\ \frac{5}{2} & \text{if} \quad \min(s, d+\sigma) = \frac{d_1}{2} \quad \text{and} \quad s = d+\sigma \\ \frac{\min(s,d+\sigma)}{d_1} & \text{if} \quad \min(s, d+\sigma) < \frac{d_1}{2} \quad \text{and} \quad s \neq d+\sigma \\ \frac{\min(s,d+\sigma)}{d_1}+1 & \text{if} \quad \min(s, d+\sigma) < \frac{d_1}{2} \quad \text{and} \quad s = d+\sigma. \end{cases} \tag{25}$$

Proposition 1. *Given $1 \leq d_1 \leq d$, and M, Q, s, σ as above, there are constants $c_1, c_2 > 0$ such that for each $n \in \mathbb{N}$ with $n \geq 2$ there is a choice of parameters m, $N^{(0)}, (N_l)_{l=0}^{m-1}$ such that the algorithm has* $\operatorname{cost}(\theta) \leq c_1 n$ *and, for each $k \in \mathcal{C}^{s,\sigma}(M, Q)$ and $f \in L_\infty(Q)$, the error satisfies*

$$e(\theta) \leq c_2 n^{-\min\left(\frac{s}{d_1}, \frac{d+\sigma}{d_1}, \frac{1}{2}\right)} (\log n)^\beta \|k\|_{\mathcal{C}^{s,\sigma}} \|f\|_{L_\infty(Q)}.$$

For the proof we need some preparations, including a number of lemmas. First note that the algorithm is bilinear in k and f, and so is the solution $T_k f$, thus, we can assume without loss of generality that

$$\|k\|_{\mathcal{C}^{s,\sigma}} \leq 1, \quad \|f\|_{L_\infty(Q)} \leq 1. \tag{26}$$

We rewrite the algorithm into a form which is convenient for our analysis. Setting for $j = 1, \ldots, N^{(0)}$,

$$\zeta_j^{(0)} = P_0 \left([g^{(0)}(x, \xi_{xj}^{(0)})]_{x \in \Gamma_0} \right), \tag{27}$$

and for $l = 0, \ldots, m-1,\ j = 1, \ldots, N_l$,

$$\zeta_{lj} = \sum_{i=1}^{n_l} (\hat{P}_{li} - P_{li}) \left([g_{li}(x, \xi_{lxj}) + h_{li}(x, \eta_{lij})]_{x \in \hat{\Gamma}_{li}} \right), \tag{28}$$

we obtain independent, $L_\infty(M)$-valued random variables, with $\zeta_j^{(0)}$ taking values in E_0 and ζ_{lj} taking values in E_{l+1}. By (12), (19), (22), and (23) we have

$$\theta = \frac{1}{N^{(0)}} \sum_{j=1}^{N^{(0)}} \zeta_j^{(0)} + \sum_{l=0}^{m-1} \frac{1}{N_l} \sum_{j=1}^{N_l} \zeta_{lj}. \tag{29}$$

Lemma 2. *The error can be estimated by*

$$e(\theta) \le \|T_k f - P_m T_k f\| + (\mathbb{E}\,\|\theta - \mathbb{E}\,\theta\|^2)^{1/2}, \tag{30}$$

with a deterministic part $\|T_k f - P_m T_k f\|$ *and a stochastic part* $(\mathbb{E}\,\|\theta - \mathbb{E}\,\theta\|^2)^{1/2}$.

Proof. From (10), (17), and (20) it follows that

$$\mathbb{E}\,\zeta_j^{(0)} = \mathbb{E}\,P_0\left([g^{(0)}(x,\xi_{xj}^{(0)})]_{x\in\Gamma_0}\right) = P_0 \int_Q k(\,.\,,y)f(y)dy = P_0 T_k f$$

and

$$\begin{aligned}\mathbb{E}\,\zeta_{lj} &= \mathbb{E}\sum_{i=1}^{n_l}(\hat{P}_{li} - P_{li})\left([g_{li}(x,\xi_{lxj}) + h_{li}(x,\eta_{lij})]_{x\in\hat{\Gamma}_{li}}\right)\\ &= \sum_{i=1}^{n_l}(\hat{P}_{li} - P_{li})\int_Q k(\,.\,,y)f(y)dy = (P_{l+1} - P_l)T_k f.\end{aligned}$$

Hence

$$\mathbb{E}\,\theta = P_m T_k f.$$

By the triangle inequality,

$$e(\theta) \le \|T_k f - P_m T_k f\| + (\mathbb{E}\,\|\theta - \mathbb{E}\,\theta\|^2)^{1/2}.$$

□

We need the following relations, which follow directly from the definitions of the a_l and b_l in (13) and (14) (recall also that we assumed $-d < \sigma$): For $l \in \mathbb{N}_0$,

$$a_l \le c\begin{cases} 2^{-(d+\sigma)l} & \text{if} \quad \sigma < 0\\ (l+1)\,2^{-dl} & \text{if} \quad \sigma = 0\\ 2^{-dl} & \text{if} \quad \sigma > 0\end{cases} \tag{31}$$

and

$$b_l \le c\begin{cases} 2^{-(d+\sigma-s)l} & \text{if} \quad \sigma - s < -d\\ l+1 & \text{if} \quad \sigma - s = -d\\ 1 & \text{if} \quad \sigma - s > -d.\end{cases} \tag{32}$$

Observe also that for $l \in \mathbb{N}_0$, $x \in M_{li}$, and $y \in C_{li} = B(x_{li},\bar{\varrho}) \setminus B(x_{li}, 2\varrho_l)$, we have

$$\begin{aligned}|x_{li} - y| &\ge 2\varrho_l\\ |x_{li} - x| &\le \varrho_l \le \frac{1}{2}|x_{li} - y|,\end{aligned}$$

and hence

$$\frac{1}{2}|x_{li}-y| \le |x-y| \le \frac{3}{2}|x_{li}-y| \quad (x \in M_{li},\ y \in C_{li}). \tag{33}$$

Finally, define

$$\alpha_0 = \begin{cases} 1 & \text{if} \quad s = d+\sigma \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

We begin with the estimate of the deterministic part in (30).

Lemma 3. *The deterministic part of the error satisfies*

$$\|T_k f - P_m T_k f\| \le c\,(m^{\alpha_0} 2^{-\min(s,d+\sigma)m} + m\,2^{-dm}). \tag{35}$$

Proof. Define the restriction operator $R_{mi} : L_\infty(M) \to L_\infty(M)$ for $f \in L_\infty(M)$ by

$$(R_{mi} f)(y) = \begin{cases} f(y) & \text{if} \quad y \in M_{mi} \\ 0 & \text{otherwise,} \end{cases}$$

and let I be the identity operator on $L_\infty(M)$. Then

$$\begin{aligned} T_k f - P_m T_k f &= (I - P_m) \int_Q k(\,.\,,y) f(y) dy \\ &= \sum_{i=1}^{n_m} (R_{mi} - P_{mi}) \int_Q k(\,.\,,y) f(y) dy. \end{aligned} \tag{36}$$

It follows from (2), (13), (26), and (31) that for $x \in M_{mi}$

$$\begin{aligned} \int_{B(x_{mi},2\varrho_m)\cap Q} k(x,y) f(y) dy &\le \int_{B(x,3\varrho_m)} \gamma_\sigma(x,y) dy \\ &= \int_{B(0,3\varrho_m)} \gamma_\sigma(0,y) dy \\ &\le c\,(2^{-(d+\sigma)m} + m\,2^{-dm}). \end{aligned} \tag{37}$$

Furthermore, from (2), (26) and (33), for $\alpha \in \mathbb{N}_0^{d_1}$, $|\alpha| \le s$, $x \in M_{mi}$,

$$\begin{aligned} \left| D_x^\alpha \int_{Q\setminus B(x_{mi},2\varrho_m)} k(x,y) f(y) dy \right| &\le \int_{Q\setminus B(x_{mi},2\varrho_m)} \gamma_{\sigma-|\alpha|}(x,y) dy \\ &\le c \int_{Q\setminus B(x_{mi},2\varrho_m)} \gamma_{\sigma-|\alpha|}(x_{mi},y) dy \\ &\le c \int_{2\varrho_m<|y|\le\bar{\varrho}} \gamma_{\sigma-|\alpha|}(0,y) dy. \end{aligned} \tag{38}$$

Using (14) and (32), we derive from (38)

$$\begin{aligned}
&\left\|\int_{Q\setminus B(x_{mi},2\varrho_m)} k(\,.\,,y)f(y)dy\right\|_{C^s(M_{mi})} \\
&\leq \; c\begin{cases} 2^{-(d+\sigma-s)m} & \text{if} \quad \sigma-s<-d \\ m & \text{if} \quad \sigma-s=-d \\ 1 & \text{if} \quad \sigma-s>-d. \end{cases}
\end{aligned} \tag{39}$$

From (37) and (5) we conclude

$$\left\|(R_{mi}-P_{mi})\int_{B(x_{mi},2\varrho_m)\cap Q} k(\,.\,,y)f(y)dy\right\|_{L_\infty(M)} \leq c\,(2^{-(d+\sigma)m}+m\,2^{-dm}),$$

while from (39) and (6) it follows that

$$\begin{aligned}
&\left\|(R_{mi}-P_{mi})\int_{Q\setminus B(x_{mi},2\varrho_m)} k(\,.\,,y)f(y)dy\right\|_{L_\infty(M)} \\
&\leq \; cm^{\alpha_0}2^{-\min(s,d+\sigma)m}.
\end{aligned} \tag{40}$$

By (36), this yields the needed estimate:

$$\|T_k f - P_m T_k f\| \leq c\,(m^{\alpha_0}2^{-\min(s,d+\sigma)m}+m\,2^{-dm}).$$

□

Now we turn to the stochastic part in (30). We need two different estimates of it.

Lemma 4. *The stochastic part of the error satisfies*

$$\begin{aligned}
&(\mathbb{E}\,\|\theta-\mathbb{E}\,\theta\|^2)^{1/2} \\
&\leq \; cm^{1/2}\Bigg(\big(N^{(0)}\big)^{-1}+ \\
&\qquad \sum_{l=0}^{m-1} N_l^{-1}\left((l+1)^{\alpha_0}2^{-\min(s,d+\sigma)l}+(l+1)\,2^{-dl}\right)^2\Bigg)^{1/2},
\end{aligned} \tag{41}$$

where α_0 was defined in (34). Furthermore,

$$\begin{aligned}
&(\mathbb{E}\,\|\theta-\mathbb{E}\,\theta\|^2)^{1/2} \\
&\leq \; c\big(N^{(0)}\big)^{-1/2}+ \\
&\qquad c\sum_{l=0}^{m-1}(l+1)^{1/2}N_l^{-1/2}\left((l+1)^{\alpha_0}2^{-\min(s,d+\sigma)l}+(l+1)\,2^{-dl}\right).
\end{aligned} \tag{42}$$

Proof. It follows from (2) and (11) that

$$\sup_{x\in\Gamma_0,\, y\in B(x,\bar\varrho)} |g^{(0)}(x,y)| \le c, \tag{43}$$

and from (2), (18), and (31) that

$$\sup_{x\in\hat\Gamma_{li},\, y\in B(x,3\varrho_l)} |g_{li}(x,y)| \le c\,(2^{-(d+\sigma)l} + (l+1)\,2^{-dl}). \tag{44}$$

Using the assumptions on k and the definition (21) of h_{li}, it is readily seen that $h_{li}(\,.\,,y) \in C^s(M_{li})$ for all $y \in C_{li}$. Moreover, (2), (32), and (33) imply

$$\begin{aligned}\sup_{y\in C_{li}} \|h_{li}(\,.\,,y)\|_{C^s(M_{li})} &\le c\,b_l \sup_{x\in M_{li},\, y\in C_{li}} \frac{\gamma_{\sigma-s}(x,y)}{\gamma_{\sigma-s}(x_{li},y)} \\ &\le c\,b_l \le c \begin{cases} 2^{-(d+\sigma-s)l} & \text{if} \quad \sigma-s<-d \\ l+1 & \text{if} \quad \sigma-s=-d \\ 1 & \text{if} \quad \sigma-s>-d, \end{cases}\end{aligned}$$

and hence, because of (7),

$$\begin{aligned}&\sup_{y\in C_{li}} \|(\hat P_{li} - P_{li})h_{li}(\,.\,,y)\|_{L_\infty(M)} \\ &\quad\le c\,2^{-sl} \sup_{y\in C_{li}} \|h_{li}(\,.\,,y)\|_{C^s(M_{li})} \le c\,(l+1)^{\alpha_0} 2^{-\min(s,d+\sigma)l}.\end{aligned} \tag{45}$$

Note that

$$c_1 2^{d_1 l} \le \dim E_l \le c_2 2^{d_1 l}. \tag{46}$$

Furthermore, the spaces E_l $(l = 0,\dots,m)$, considered in the norm of $L_\infty(M)$, are uniformly ismorphic to $\ell_\infty^{\dim E_l}$ in the sense that there exist linear isomorphisms $U_l : \ell_\infty^{\dim E_l} \to E_l$ with

$$\|U_l\|\,\|U_l^{-1}\| \le c,$$

where c is independent of l and m. This is readily checked by identifying $\ell_\infty^{\dim E_l}$ with $\ell_\infty\,({}^*\!\cup_{i=1}^{n_l}\Gamma_{li})$, where ${}^*\!\cup$ stands for the disjoint union, and setting

$$U_l v = \sum_{i=1}^{n_l} P_{li}(v\,|_{\Gamma_{li}}).$$

By (27), (43) and (5),

$$\sup_{\omega\in\Omega} \|\zeta_j^{(0)}(\omega)\|_{L_\infty(M)} \le c. \tag{47}$$

Moreover, by (28), (44), (5), and (45)

$$\sup_{\omega\in\Omega} \|\zeta_{lj}(\omega)\|_{L_\infty(M)} \le c\,((l+1)^{\alpha_0} 2^{-\min(s,d+\sigma)l} + (l+1)\,2^{-dl}). \tag{48}$$

Now the first estimate (41) follows from Lemma 1, (29), (46), (47), and (48):

$$\begin{aligned}
& (\mathbb{E}\,\|\theta - \mathbb{E}\,\theta\|^2)^{1/2} = \mathrm{Var}(\theta)^{1/2}_{L_\infty(M)} = \mathrm{Var}(\theta)^{1/2}_{E_m} \\
\le\ & cm^{1/2} \left(\left(N^{(0)}\right)^{-2} \sum_{j=1}^{N^{(0)}} \mathrm{Var}\left(\zeta_j^{(0)}\right)_{E_m} + \sum_{l=0}^{m-1} N_l^{-2} \sum_{j=1}^{N_l} \mathrm{Var}(\zeta_{lj})_{E_m} \right)^{1/2} \\
\le\ & cm^{1/2} \Bigg(\left(N^{(0)}\right)^{-1} + \\
& \sum_{l=0}^{m-1} N_l^{-1} \left((l+1)^{\alpha_0} 2^{-\min(s,d+\sigma)l} + (l+1)\,2^{-dl} \right)^2 \Bigg)^{1/2}.
\end{aligned}$$

Here we used that

$$\mathrm{Var}(\zeta)_{E_m} = \mathrm{Var}(\zeta)_{L_\infty(M)} \le 4\mathbb{E}\,\|\zeta\|^2_{L_\infty(M)} \le 4 \sup_{\omega\in\Omega} \|\zeta(\omega)\|^2_{L_\infty(M)}$$

for a random variable ζ on (Ω, Σ, μ) with values in $E_m \subset L_\infty(M)$. Applying first the triangle inequality and then Lemma 1 for each l separately gives the desired second estimate (42):

$$
\begin{aligned}
&(\mathbb{E}\,\|\theta-\mathbb{E}\,\theta\|^2)^{1/2} = \mathrm{Var}(\theta)^{1/2}_{L_\infty(M)}\\
\le\ & \mathrm{Var}\left(\frac{1}{N^{(0)}}\sum_{j=1}^{N^{(0)}}\zeta_j^{(0)}\right)^{1/2}_{L_\infty(M)} + \sum_{l=0}^{m-1}\mathrm{Var}\left(\frac{1}{N_l}\sum_{j=1}^{N_l}\zeta_{lj}\right)^{1/2}_{L_\infty(M)}\\
=\ & \mathrm{Var}\left(\frac{1}{N^{(0)}}\sum_{j=1}^{N^{(0)}}\zeta_j^{(0)}\right)^{1/2}_{E_0} + \sum_{l=0}^{m-1}\mathrm{Var}\left(\frac{1}{N_l}\sum_{j=1}^{N_l}\zeta_{lj}\right)^{1/2}_{E_{l+1}}\\
\le\ & c\left(\big(N^{(0)}\big)^{-2}\sum_{j=1}^{N^{(0)}}\mathrm{Var}\big(\zeta_l^{(0)}\big)_{E_0}\right)^{1/2} +\\
& c\sum_{l=0}^{m-1}(l+1)^{1/2}\left(N_l^{-2}\sum_{j=1}^{N_l}\mathrm{Var}\left(\zeta_{lj}\right)_{E_{l+1}}\right)^{1/2}\\
\le\ & c\big(N^{(0)}\big)^{-1/2} +\\
& c\sum_{l=0}^{m-1}(l+1)^{1/2}N_l^{-1/2}\left((l+1)^{\alpha_0}2^{-\min(s,d+\sigma)l} + (l+1)\,2^{-dl}\right).
\end{aligned}
$$

□

Proof of Proposition 1. It remains to provide the choice of parameters and to derive the final error estimates. Let $n\in\mathbb{N}$ with $n\ge 2$ be given. First assume that $\min(s,d+\sigma)>d_1/2$. Choose any $\tau>0$ such that

$$\min(s,d+\sigma,d) > (d_1+\tau)/2,$$

and let (recall that log always means $\log_2$)

$$m = \left\lceil \frac{\log n}{d_1+\tau}\right\rceil,$$

$$N^{(0)} = n, \quad N_l = \left\lceil n\,2^{-(d_1+\tau)l}\right\rceil \quad (l=0,\dots,m-1).$$

Then the cost is bounded by

$$
\begin{aligned}
\mathrm{cost}(\theta) &= N^{(0)} + \sum_{l=0}^{m-1} n_l N_l \le n + \sum_{l=0}^{m-1} 2^{d_1 l}(n2^{-(d_1+\tau)l}+1)\\
&\le c(n+2^{d_1 m}) \le c\,n.
\end{aligned}
$$

We estimate the stochastic error by Lemma 4, (42):

$$\begin{aligned}
&(\mathbb{E}\,\|\theta - \mathbb{E}\,\theta\|^2)^{1/2} \\
&\quad\le\; c\,n^{-1/2} + \\
&\qquad c\sum_{l=0}^{m-1}(l+1)^{1/2}n^{-1/2}2^{(d_1+\tau)l/2}\left((l+1)^{\alpha_0}2^{-\min(s,d+\sigma)l} + (l+1)2^{-dl}\right) \\
&\quad\le\; cn^{-1/2}.
\end{aligned}$$

By Lemma 3,

$$\|T_kf - P_mT_kf\| \le c\,(m^{\alpha_0}2^{-\min(s,d+\sigma)m} + m\,2^{-dm}) \le c\,2^{-(d_1+\tau)m/2} \le cn^{-1/2}.$$

Now the desired result follows from Lemma 2.

Next assume $\min(s, d+\sigma) = d_1/2$. We put

$$m = \left\lceil \frac{\log n}{d_1} \right\rceil, \tag{49}$$

and

$$N^{(0)} = n, \quad N_l = \left\lceil nm^{-1}2^{-d_1 l} \right\rceil \quad (l = 0, \dots, m-1).$$

Then the cost can be estimated by

$$N^{(0)} + \sum_{l=0}^{m-1} n_l N_l \le n + \sum_{l=0}^{m-1} 2^{d_1 l}(nm^{-1}2^{-d_1 l} + 1) \le c(n + 2^{d_1 m}) \le c\,n.$$

By Lemma 4, (41), the stochastic error satisfies

$$\begin{aligned}
&(\mathbb{E}\,\|\theta - \mathbb{E}\,\theta\|^2)^{1/2} \\
&\quad\le\; cm^{1/2}\Bigg(n^{-1} + \\
&\qquad \sum_{l=0}^{m-1} n^{-1}m2^{d_1 l}\left((l+1)^{2\alpha_0}2^{-2\min(s,d+\sigma)l} + (l+1)^2 2^{-2dl}\right)\Bigg)^{1/2} \\
&\quad\le\; cmn^{-1/2}\left(\sum_{l=0}^{m-1}(l+1)^{2\alpha_0}\right)^{1/2} \\
&\quad\le\; cn^{-1/2}m^{\alpha_0+3/2} \le cn^{-1/2}(\log n)^{\alpha_0+3/2}.
\end{aligned}$$

Furthermore, by Lemma 3,

$$\|T_kf - P_mT_kf\| \le c(m^{\alpha_0}2^{-d_1 m/2} + m2^{-dm}) \le cn^{-1/2}(\log n)^{\alpha_0}.$$

An application of Lemma 2 concludes the proof in this case.

Finally, we assume $\min(s, d+\sigma) < d_1/2$. Choose any τ with

$$0 < \tau < d_1 - 2\min(s, d+\sigma),$$

and put

$$m = \left\lceil \frac{\log n - \log\log n}{d_1} \right\rceil, \tag{50}$$

$$N^{(0)} = n, \quad N_l = \left\lceil n\, 2^{-d_1 l - \tau(m-l)} \right\rceil \quad (l = 0, \dots, m-1).$$

Note that (50) implies

$$(n/\log n)^{1/d_1} \le 2^m \le 2(n/\log n)^{1/d_1}.$$

The cost is bounded by

$$\begin{aligned} N^{(0)} + \sum_{l=0}^{m-1} n_l N_l &\le n + \sum_{l=0}^{m-1} 2^{d_1 l}(n 2^{-d_1 l - \tau(m-l)} + 1) \\ &\le c\left(n \sum_{l=0}^{m-1} 2^{-\tau(m-l)} + 2^{d_1 m}\right) \le c\, n. \end{aligned}$$

Relation (41) of Lemma 4 gives

$$\begin{aligned} &(\mathbb{E}\,\|\theta - \mathbb{E}\,\theta\|^2)^{1/2} \\ \le\;& cm^{1/2}\Bigg(n^{-1} + \\ &\sum_{l=0}^{m-1} n^{-1} 2^{d_1 l + \tau(m-l)}\Big((l+1)^{2\alpha_0} 2^{-2\min(s,d+\sigma)l} + (l+1)^2 2^{-2dl}\Big)\Bigg)^{1/2} \\ \le\;& cm^{1/2}\left(n^{-1} + \sum_{l=0}^{m-1} n^{-1} 2^{d_1 l + \tau(m-l)}(l+1)^{2\alpha_0} 2^{-2\min(s,d+\sigma)l}\right)^{1/2} \\ \le\;& cn^{-1/2} m^{1/2+\alpha_0}\left(2^{\tau m} \sum_{l=0}^{m-1} 2^{(d_1 - \tau - 2\min(s,d+\sigma))l}\right)^{1/2} \\ \le\;& cn^{-1/2} m^{1/2+\alpha_0} 2^{(d_1/2 - \min(s,d+\sigma))m} \\ \le\;& cn^{-1/2} (\log n)^{1/2+\alpha_0} (n/\log n)^{(d_1/2 - \min(s,d+\sigma))/d_1} \\ \le\;& cn^{-\min(s,d+\sigma)/d_1} (\log n)^{\min(s,d+\sigma)/d_1 + \alpha_0}. \end{aligned}$$

Moreover, using Lemma 3 again,

$$\begin{aligned}\|T_k f - P_m T_k f\| &\le c\,(m^{\alpha_0} 2^{-\min(s,d+\sigma)m} + m\,2^{-dm}) \le c\,m^{\alpha_0} 2^{-\min(s,d+\sigma)m} \\ &\le cn^{-\min(s,d+\sigma)/d_1} (\log n)^{\alpha_0+\min(s,d+\sigma)/d_1}.\end{aligned}$$

A final application of Lemma 2 completes the proof. □

4 Lower bounds and complexity

We shall be concerned with the information complexity exclusively, that is, we only count information operations. This makes the lower bound statements stronger. The upper bounds obtained in the previous section were anyway accompanied by estimates of the total cost, including arithmetic operations and random variable generation.

First we describe the needed notions in a general framework. We refer to [19] and [17] for further background on the theory of information-based complexity. A numerical problem is given by a tuple $\mathcal{P} = (F, G, S, K, \Lambda)$, where F is a non-empty set, G a normed space over $\mathbb{K}$, where $\mathbb{K}$ stands for the set of real or complex numbers, S a mapping from F to G, K a non-empty set and Λ a non-empty set of mappings from F to K. We seek to compute (approximately) $S(f)$ for $f \in F$ using information about $f \in F$ of the form $\lambda(f)$ for $\lambda \in \Lambda$.

Usually F is a set in a function space, S is the solution operator, mapping the input $f \in F$ to the exact solution $S(f)$ of our problem, which we want to approximate. Λ is usually a set of linear functionals, and K is mostly $\mathbb{R}$ or $\mathbb{C}$ (however, for understanding the complexity under certain more powerful information assumptions, like, e.g., in [8], it is convenient to keep K general). G is usually a space containing both the solutions and the approximations, and it is equipped with a norm, in which the error is measured. (Compare also the specifications to our situation given before Propositions 2 and 3.)

Let $k^* = K$. (We want $\{k^*\}$ to be any one-element set such that $k^* \notin K$. With the choice $k^* = K$, this is the case, since a set never contains itself as an element.) We use this to define the zero-th power of K as $K^0 = \{k^*\}$. In the sequel it will be convenient to consider $f \in F$ also as a function on Λ with values in K by setting $f(\lambda) := \lambda(f)$. Let $\mathcal{F}(\Lambda, K)$ denote the set of all functions from Λ to K.

A deterministic algorithm A for $\mathcal{P}$ is a tuple

$$A = ((L_i)_{i=1}^\infty, (\tau_i)_{i=0}^\infty, (\varphi_i)_{i=0}^\infty)$$

where for each i,

$$\begin{aligned} L_i &: K^{i-1} \to \Lambda \\ \tau_i &: K^i \to \{0,1\} \\ \varphi_i &: K^i \to G \end{aligned}$$

are any mappings. Given $f \in \mathcal{F}(\Lambda, K)$, we associate with it a sequence $(z_i)_{i=0}^{\infty}$ with $z_i \in K^i$, we call it the computational sequence of A at input f, defined as follows:

$$\begin{aligned} z_0 &= k^* \\ z_i &= (f(L_1(z_0)), \ldots, f(L_i(z_{i-1}))) \quad (i \geq 1). \end{aligned}$$

Let the cardinality $\operatorname{card}(A, f)$ of A at input f be the first integer $n \geq 0$ with $\tau_n(z_n) = 1$, and put $\operatorname{card}(A, f) = +\infty$ if there is no such n. Define

$$\operatorname{Dom}(A) = \{f \in \mathcal{F}(\Lambda, K) : \operatorname{card}(A, f) < \infty\}.$$

For $f \in \operatorname{Dom}(A)$ and $n = \operatorname{card}(A, f)$ we define the output $A(f)$ of algorithm A at input f as

$$A(f) = \varphi_n(z_n).$$

Let $\mathcal{A}^{\det}(\mathcal{P})$ be the set of all deterministic algorithms for $\mathcal{P}$. If $\mathcal{P}$ is fixed, we write shortly $\mathcal{A}^{\det}$. For $A \in \mathcal{A}^{\det}$ define

$$\operatorname{card}(A, F) = \sup_{f \in F} \operatorname{card}(A, f),$$

and the error of A as

$$e(S, A, F) = \sup_{f \in F} \|S(f) - A(f)\|_G$$

if $F \subseteq \operatorname{Dom}(A)$, and $e(S, A, F) = +\infty$ otherwise. Furthermore, for $n \in \mathbb{N}_0$, let the n-th deterministic minimal error be defined as

$$e_n^{\det}(S, F) = \inf\{e(S, A, F) : A \in \mathcal{A}^{\det},\ \operatorname{card}(A, F) \leq n\}.$$

The meaning of this crucial quantity of information-based complexity is the following: No deterministic algorithm that uses at most n informations on f can provide a smaller error than $e_n^{\det}(S, F)$.

A randomized (or Monte Carlo) algorithm for $\mathcal{P}$

$$A = ((\Omega, \Sigma, \mu),\ (A_\omega)_{\omega \in \Omega}),$$

consists of a probability space (Ω, Σ, μ), and a family

$$A_\omega \in \mathcal{A}^{\rm det}(\mathcal{P}) \quad (\omega \in \Omega).$$

Define $\mathrm{Dom}(A)$ to be the set of all $f \in \mathcal{F}(\Lambda, K)$ such that $\mathrm{card}(A_\omega, f)$ is a measurable function of ω,

$$\mathrm{card}(A_\omega, f) < \infty \text{ for almost all } \omega \in \Omega,$$

and $A_\omega(f)$ is a G-valued random variable, meaning that $A_\omega(f)$ is Borel measurable and there is a separable subspace G_0 of G (which may depend on f) such that

$$A_\omega(f) \in G_0 \text{ for almost all } \omega \in \Omega.$$

Let $\mathcal{A}^{\rm ran}(\mathcal{P})$, or shortly $\mathcal{A}^{\rm ran}$ denote the class of all randomized algorithms for $\mathcal{P}$. Given $A \in \mathcal{A}^{\rm ran}$ and $f \in \mathcal{F}(\Lambda, K)$, define

$$\mathrm{card}(A, f) = \int_\Omega \mathrm{card}(A_\omega, f) \, d\mu(\omega)$$

if $f \in \mathrm{Dom}(A)$ and $\mathrm{card}(A, f) = +\infty$ otherwise. Put

$$\mathrm{card}(A, F) = \sup_{f \in F} \mathrm{card}(A, f).$$

The error of $A \in \mathcal{A}^{\rm ran}$ is given by

$$e(S, A, F) = \sup_{f \in F} \int_\Omega \|S(f) - A_\omega(f)\|_G \, d\mu(\omega).$$

if $F \subseteq \mathrm{Dom}(A)$, and $e(S, A, F) = +\infty$ otherwise. We have chosen the first moment, that is, the $L_1(\Omega, \mu)$ norm for the error. Clearly, we could have considered the error also in the sense of $L_p(\Omega, \mu)$, $1 < p < \infty$, which would not cause essential changes. For $n \in \mathbb{N}_0$ the n-th randomized minimal error is defined as

$$e_n^{\rm ran}(S, F) = \inf\{e(S, A, F) : A \in \mathcal{A}^{\rm ran}, \ \mathrm{card}(A, F) \le n\}.$$

Hence, no randomized algorithm that uses (on the average) at most n information functionals can provide a smaller error than $e_n^{\rm ran}(S, F)$.

We shall reduce the lower estimate of the minimal randomized error in the usual way to the average case setting. We only need measures whose support is a finite set. So let ν be such a measure on F, let $A \in \mathcal{A}^{\rm det}$. Put

$$\begin{aligned}
\mathrm{card}(A, \nu) &= \int_F \mathrm{card}(A, f) \, d\nu(f), \\
e(S, A, \nu) &= \int_F \|S(f) - A(f)\|_G \, d\nu(f), \\
e_n^{\rm avg}(S, \nu) &= \inf\{e(S, A, \nu) : A \in \mathcal{A}^{\rm det}, \mathrm{card}(A, \nu) \le n\}.
\end{aligned}$$

Lemma 5. *For each probability measure ν on F of finite support and each $n \in \mathbb{N}$,*

$$e_n^{\mathrm{ran}}(S,F) \geq \frac{1}{2}\, e_{2n}^{\mathrm{avg}}(S,\nu).$$

This is well-known, and can be found, for example, in [5]. Although dealing with a slightly less general setting, the proof of Lemma 2 in there literally carries over.

Next we consider problems $\mathcal{P}$ which are linear in the sense that $K = \mathbb{K}$ (the set of real or complex numbers), F is a subset of a linear space X over $\mathbb{K}$, S is the restriction to F of a linear operator from X to G, and all mappings $\lambda \in \Lambda$ are restrictions to F of linear mappings from X to $\mathbb{K}$.

Lemma 6. *Let $n, \bar{n} \in \mathbb{N}$ with $\bar{n} > 2n$, assume that there are $(f_i)_{i=1}^{\bar{n}} \subseteq F$ such that the sets $\{\lambda \in \Lambda : f_i(\lambda) \neq 0\}$ $(i = 1, \dots, \bar{n})$ are mutually disjoint, and for all sequences $(\alpha_i)_{i=1}^{\bar{n}} \in \{-1, 1\}^{\bar{n}}$ we have $\sum_{i=1}^{\bar{n}} \alpha_i f_i \in F$. Define the measure ν on F to be the distribution of $\sum_{i=1}^{\bar{n}} \varepsilon_i f_i$, where ε_i are independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_i = 1\} = \mathbb{P}\{\varepsilon_i = -1\} = 1/2$. Then*

$$e_n^{\mathrm{avg}}(S,\nu) \geq \frac{1}{2} \min_I \mathbb{E}\, \| \sum_{i \in I} \varepsilon_i S(f_i) \|_G,$$

where the minimum is taken over all subsets I of $\{1, \dots, \bar{n}\}$ with $|I| \geq \bar{n} - 2n$.

The proof follows the lines of the lower bound proof in [5], pp. 170-173. We omit it here.

Corollary 1. *There is a constant $c > 0$ such that if G is a Hilbert space, then under the assumptions of Lemma 6,*

$$e_n^{\mathrm{avg}}(S,\nu) \geq c \min_I \left(\sum_{i \in I} \|S(f_i)\|_G^2 \right)^{1/2},$$

the minimum taken over all subsets I of $\{1, \dots, \bar{n}\}$ with $|I| \geq \bar{n} - 2n$.

Proof. This is a direct consequence of the generalized parallelogram identity

$$\mathbb{E}\, \| \sum_{i=1}^{m} \varepsilon_i u_i \|_G^2 = \sum_{i=1}^{m} \|u_i\|_G^2$$

for elements u_i in a Hilbert space G, and the equivalence of moments, see [13], Theorem 4.7, which asserts the existence of an absolute constant $c > 0$ with

$$\mathbb{E}\,\|\sum_{i=1}^{m}\varepsilon_i u_i\|_G \ge c\left(\mathbb{E}\,\|\sum_{i=1}^{m}\varepsilon_i u_i\|_G^2\right)^{1/2}.$$

□

An important tool for lower bound proofs is reduction. We need a simple result, which is a special case of Proposition 1 in [9].

Let $\widetilde{\mathcal{P}} = (\widetilde{F}, \widetilde{G}, \widetilde{S}, \widetilde{K}, \widetilde{\Lambda})$ be another numerical problem. Assume that $R : F \to \widetilde{F}$ is a mapping such that there exist mappings $\eta : \widetilde{\Lambda} \to \Lambda$ and $\varrho : \widetilde{\Lambda} \times K \to \widetilde{K}$ with

$$(R(f))(\widetilde{\lambda}) = \varrho(\widetilde{\lambda}, f(\eta(\widetilde{\lambda}))) \tag{51}$$

for all $f \in F$ and $\widetilde{\lambda} \in \widetilde{\Lambda}$. Suppose that $L : \widetilde{G} \to G$ is a Lipschitz mapping, that is, there is a constant $c \ge 0$ such that

$$\|L(x) - L(y)\|_G \le c\,\|x - y\|_{\widetilde{G}} \quad \text{for all} \quad x, y \in \widetilde{G}.$$

The Lipschitz constant $\|L\|_{\mathrm{Lip}}$ is the smallest constant c such that the relation above holds. Finally, assume that

$$S = L \circ \widetilde{S} \circ R.$$

Lemma 7. *For all $n \in \mathbb{N}_0$,*

$$e_n^{\mathrm{ran}}(S, F) \quad \le \quad \|L\|_{\mathrm{Lip}}\, e_n^{\mathrm{ran}}(\widetilde{S}, \widetilde{F}). \tag{52}$$

Now we return to the concrete numerical problems studied before. Let M and Q be as defined in the beginning, including the case $d_1 = 0$. We assume, additionally, that Q has non-empty interior. Our first result concerns integral operators with a fixed, weakly singular kernel $k \in \mathcal{C}^{s,\sigma}(M, Q)$. Let $\mathcal{L}_\infty(Q)$ be the linear space of all Lebesgue measurable essentially bounded real-valued functions on Q, equipped with the seminorm

$$|f|_{\mathcal{L}_\infty} = \operatorname{ess\,sup}_{y\in Q}|f(y)|.$$

Note that the space $\mathcal{L}_\infty(Q)$ consists of functions defined everywhere on Q. In contrast, the space $L_\infty(Q)$ consists of equivalence classes, being the quotient of $\mathcal{L}_\infty(Q)$ over the subspace $\{|f|_{\mathcal{L}_\infty} = 0\}$. The reason for this distinction is that in $\mathcal{L}_\infty(Q)$ function values are defined, while they are not in $L_\infty(Q)$.

As a target space, we still use the normed space $L_\infty(M)$. So we consider T_k as an operator from $\mathcal{L}_\infty(Q)$ to $L_\infty(M)$ (note that T_k is defined correctly on both $\mathcal{L}_\infty(Q)$ and $L_\infty(Q)$, we therefore use the same notation T_k in both cases). For the following proposition we set

$$F = \mathcal{B}_{\mathcal{L}_\infty(Q)} = \{f \in \mathcal{L}_\infty(Q) \, : \, |f|_{\mathcal{L}_\infty} \le 1\},$$

$$G = \begin{cases} L_\infty(M) & \text{if} \quad d_1 \ge 1 \\ \mathbb{R} & \text{if} \quad d_1 = 0, \end{cases}$$

$S = T_k$, and $\Lambda = \{\delta_y \, : \, y \in Q\}$, where $\delta_y(f) = f(y)$ for $f \in \mathcal{B}_{\mathcal{L}_\infty(Q)}$. Throughout the rest of this section and also in the next section we will have $K = \mathbb{K} = \mathbb{R}$, so we do not repeat this assumption.

Define

$$\alpha_1 = \begin{cases} \frac{1}{2} & \text{if} \quad d_1 = d = -2\sigma \\ 0 & \text{otherwise.} \end{cases} \tag{53}$$

Proposition 2. *Let $0 \le d_1 \le d$, assume that Q has non-empty interior, and let $k \in \mathcal{C}^{s,\sigma}(M,Q)$. Depending on the parameters, we make the following further assumptions about k:*

1. *If $\sigma \le d_1/2 - d$ (which implies $d_1 \ne 0$), we suppose that there exist $x_0 \in M^0 \cap Q^0$, $\delta_0 > 0$ and $\vartheta_0 \ne 0$ such that $\vartheta_0 k(x,y) \ge |x-y|^\sigma$ for all $x \in M$ and $y \in Q$ with $|x - x_0| \le \delta_0$, $|y - x_0| \le \delta_0$, and $x \ne y$.*

2. *If $\sigma > d_1/2 - d$ and $d_1 \ge 1$, we assume that there exist $x_0 \in M^0$, $y_0 \in Q^0$, $\delta_0 > 0$ and $\vartheta_0 \ne 0$ such that $\vartheta_0 k(x,y) \ge 1$ for all $x \in M$ and $y \in Q$ with $|x - x_0| \le \delta_0$, $|y - y_0| \le \delta_0$, and $x \ne y$.*

3. *If $d_1 = 0$, we suppose that there exist $y_0 \in Q^0$, $\delta_0 > 0$ and $\vartheta_0 \ne 0$ such that $\vartheta_0 k(0,y) \ge 1$ for all $y \in Q$ with $|y - y_0| \le \delta_0$ and $y \ne 0$.*

Then there is a constant $c > 0$ (depending on k) such that for all $n \in \mathbb{N}$ with $n \ge 2$,

$$e_n^{\mathrm{ran}}(T_k, \mathcal{B}_{\mathcal{L}_\infty(Q)}) \ge c n^{-\min\left(\frac{d+\sigma}{d_1}, \frac{1}{2}\right)} (\log n)^{\alpha_1}$$

(with $\frac{d+\sigma}{d_1}$ interpreted as $+\infty$ for $d_1 = 0$).

Proof. Case 1: Since x_0 is an inner point of Q, we can find a cube $Q' = x_0 + \delta_1[-1/2, 1/2]^d$ contained in Q. By choosing $\delta_1 > 0$ small enough, we may assume that $|y - x_0| \le \delta_0$ for all $y \in Q'$, and, since x_0 is also an inner point of M, that $M' = x_0 + \delta_1[-1/2, 1/2]^{d_1} \times \{0^{(d-d_1)}\}$ is contained in M.

It follows that $M' \subseteq Q'$ and $\vartheta_0 k(x,y) \geq |x-y|^\sigma$ for all $x \in M'$ and $y \in Q'$ with $x \neq y$. Let $n \in \mathbb{N}$, $n \geq 2$. Set

$$m = \left\lceil \frac{\log n}{d_1} \right\rceil + \left\lceil \log \sqrt{d} \right\rceil + 3. \tag{54}$$

Let $\{Q_i',\, i = 1, \ldots, 2^{dm}\}$ be the canonical decomposition of Q' into closed subcubes of sidelength $2^{-m}\delta_1$. Let ψ be a continuous function on $\mathbb{R}^d$ with $\operatorname{supp}\psi \subseteq Q'$ and $0 < \psi(y) \leq 1$ for all y in the interior of Q'. Let ψ_i be the function obtained by shrinking ψ to Q_i', i.e.,

$$\psi_i(y) = \psi(x_0 + 2^m(y - y_i)),$$

with y_i the center of Q_i'. Fix $1 \leq i \leq 2^{dm}$ and let $x \in M'$ satisfy

$$|x - y_i| \geq \sqrt{d}\, 2^{-m}\delta_1. \tag{55}$$

Observe that for all $y \in Q_i'$,

$$|y_i - y| \leq \sqrt{d}\, 2^{-m-1}\delta_1.$$

Therefore,

$$|x-y| \leq |x-y_i| + |y_i - y| \leq \frac{3}{2}|x - y_i|.$$

Since, by assumption of case 1, $\sigma < 0$, we get

$$\begin{aligned} |(T_k\psi_i)(x)| &\geq |\vartheta_0|^{-1} \int_{Q'} |x-y|^\sigma \psi_i(y)dy \\ &\geq |\vartheta_0|^{-1} \left(\frac{3}{2}|x-y_i|\right)^\sigma \int_{Q'} \psi_i(y)dy \\ &\geq c2^{-dm}|x-y_i|^\sigma, \end{aligned} \tag{56}$$

provided (55) holds (the constants appearing in this proof may depend on k). Define $J : L_\infty(M) \to L_2(M')$ by $Jf = f|_{M'}$. Let

$$I_m = \{1 \leq i \leq 2^{dm} \,:\, Q_i' \cap M' \neq \emptyset\}.$$

Then

$$|I_m| = 2^{d_1 m + d - d_1},$$

and therefore, by (54),

$$|I_m| \geq 8n. \tag{57}$$

By (56) we have for $i \in I_m$

$$\|JT_k\psi_i\|^2_{L_2(M')} \geq c2^{-2dm} \int_{\left\{x\in M' \,:\, |x-y_i|\geq\sqrt{d}\,2^{-m}\delta_1\right\}} |x-y_i|^{2\sigma}dx. \qquad (58)$$

Let y_i' be the orthogonal projection of y_i onto $\mathbb{R}^{d_1} \times \{0^{(d-d_1)}\}$. Clearly, $y_i' \in M'$, and for all $x \in M'$,

$$|x - y_i'| \leq |x - y_i|. \qquad (59)$$

Since $i \in I_m$, it follows that

$$|y_i' - y_i| = \sqrt{d-d_1}2^{-m-1}\delta_1 < \sqrt{d}2^{-m-1}\delta_1.$$

Therefore, under asssumption (55)

$$\sqrt{d}2^{-m}\delta_1 \leq |x-y_i| \leq |x-y_i'| + |y_i'-y_i| \leq |x-y_i'| + \sqrt{d}2^{-m-1}\delta_1,$$

which, in turn, implies

$$|x - y_i| \leq 2|x - y_i'|. \qquad (60)$$

From relations (59) and (60) we get

$$\begin{aligned}
&\int_{\left\{x\in M' \,:\, |x-y_i|\geq\sqrt{d}\,2^{-m}\delta_1\right\}} |x-y_i|^{2\sigma}dx \\
\geq\ & 2^{2\sigma}\int_{\left\{x\in M' \,:\, |x-y_i'|\geq\sqrt{d}\,2^{-m}\delta_1\right\}} |x-y_i'|^{2\sigma}dx \\
\geq\ & 2^{2\sigma}\int_{\left\{x\in M' \,:\, \sqrt{d}\,2^{-m}\delta_1\leq|x-y_i'|\leq 2^{-1}\delta_1\right\}} |x-y_i'|^{2\sigma}dx. \qquad (61)
\end{aligned}$$

Since $2^{-1}\delta_1$ is half the side length of M', at least one (d_1-dimensional) quadrant of the ball

$$\left\{x \in \mathbb{R}^{d_1} \,:\, |x - y_i'| \leq 2^{-1}\delta_1\right\}$$

fully belongs to M'. This gives

$$\begin{aligned}
&\int_{\left\{x\in M' \,:\, \sqrt{d}\,2^{-m}\delta_1\leq|x-y_i'|\leq 2^{-1}\delta_1\right\}} |x-y_i'|^{2\sigma}dx \\
\geq\ & 2^{-d_1}\int_{\left\{x\in \mathbb{R}^{d_1} \,:\, \sqrt{d}\,2^{-m}\delta_1\leq|x-y_i'|\leq 2^{-1}\delta_1\right\}} |x-y_i'|^{2\sigma}dx. \qquad (62)
\end{aligned}$$

By (54),

$$2\sqrt{d}2^{-m}\delta_1 \leq 2^{-1}\delta_1.$$

Therefore

$$\int_{\left\{x\in\mathbb{R}^{d_1}\,:\,\sqrt{d}\,2^{-m}\delta_1\leq|x-y_i'|\leq 2^{-1}\delta_1\right\}} |x-y_i'|^{2\sigma}dx \geq c\,2^{-(d_1+2\sigma)m}m^{2\alpha_1}, \tag{63}$$

where $\alpha_1 = 1/2$ if $\sigma = -d_1/2$ (which, because of $\sigma \leq d_1/2 - d$, can only happen if we also have $d_1 = d$) and $\alpha_1 = 0$ otherwise. Joining (58) with (61–63), we obtain

$$\|JT_k\psi_i\|^2_{L_2(M')} \geq c2^{-(2d+d_1+2\sigma)m}m^{2\alpha_1}.$$

Using Lemma 5, Corollary 1, and (57), we get

$$e_n^{\mathrm{ran}}(JT_k, \mathcal{B}_{\mathcal{L}_\infty(Q)})^2 \geq c\,n2^{-(2d+2\sigma+d_1)m}m^{2\alpha_1}, \tag{64}$$

Since $\|J\| \leq 1$, a simple consequence of Lemma 7 is

$$e_n^{\mathrm{ran}}(T_k, \mathcal{B}_{\mathcal{L}_\infty(Q)}) \geq e_n^{\mathrm{ran}}(JT_k, \mathcal{B}_{\mathcal{L}_\infty(Q)}),$$

which together with (54) and (64) gives,

$$e_n^{\mathrm{ran}}(T_k, \mathcal{B}_{\mathcal{L}_\infty(Q)}) \geq cn^{-(d+\sigma)/d_1}(\log n)^{\alpha_1}.$$

Case 2: Here we argue similarly. We put $Q' = y_0 + \delta_1[-1/2, 1/2]^d$ and $M' = x_0 + \delta_1[-1/2, 1/2]^{d_1} \times \{0^{(d-d_1)}\}$. We choose $\delta_1 > 0$ so small that $M' \subseteq M$, $Q' \subseteq Q$, and $\vartheta_0 k(x, y) \geq 1$ for all $x \in M'$ and $y \in Q'$ with $x \neq y$. Let $n \in \mathbb{N}$, put

$$m = \left\lceil \frac{\log n}{d} \right\rceil + 3, \tag{65}$$

and let ψ_i $(i = 1, \dots, 2^{dm})$ be defined as above. Then for $x \in M'$,

$$|(T_k\psi_i)(x)| \geq |\vartheta_0|^{-1}\int_{Q'}\psi_i(y)dy \geq c\,2^{-dm}$$

and hence, for $i = 1, \dots, 2^{dm}$,

$$\|JT_k\psi_i\|^2_{L_2(M')} \geq c\,2^{-2dm}.$$

Using (65), it follows as in the proof of (i) that

$$e_n^{\mathrm{ran}}(T_k, \mathcal{B}_{\mathcal{L}_\infty(Q)}) \geq e_n^{\mathrm{ran}}(JT_k, \mathcal{B}_{\mathcal{L}_\infty(Q)}) \geq c\,2^{-dm/2} \geq cn^{-1/2}.$$

The same argument can be used for the case 3, with $L_2(M')$ replaced by $\mathbb{R}$. □

Note that the case $d_1 = 0$ is essentially the known lower bound for integration.

The following theorem summarizes our results for the case of a single, fixed operator and shows that upper and lower bounds are matching, up to logarithmic factors.

Theorem 1. *Let $0 \le d_1 \le d$, let $\sigma \in \mathbb{R}$, $-d < \sigma < +\infty$, let M, Q be as defined in section 2, assume that Q has non-empty interior, and let $s \in \mathbb{N}$ be such that $\frac{s}{d_1} \ge \min\left(\frac{d+\sigma}{d_1}, \frac{1}{2}\right)$. Then there is a constant $c_1 > 0$ such that for all $k \in \mathcal{C}^{s,\sigma}(M, Q)$ and $n \in \mathbb{N}$ with $n \ge 2$,*

$$e_n^{\mathrm{ran}}(T_k, \mathcal{B}_{\mathcal{L}_\infty(Q)}) \le c_1 \|k\|_{\mathcal{C}^{s,\sigma}} n^{-\min\left(\frac{d+\sigma}{d_1}, \frac{1}{2}\right)} (\log n)^{\beta}.$$

Moreover, for each $k \in \mathcal{C}^{s,\sigma}(M, Q)$ satisfying the assumptions of Proposition 2 there is a constant $c_2 > 0$ (which may depend on k) such that for all $n \in \mathbb{N}$ with $n \ge 2$,

$$c_1 n^{-\min\left(\frac{d+\sigma}{d_1}, \frac{1}{2}\right)} (\log n)^{\alpha_1} \le e_n^{\mathrm{ran}}(T_k, \mathcal{B}_{\mathcal{L}_\infty(Q)}).$$

The constants α_1 and β were defined in (53) and (25), respectively,

Proof. The lower bound is a consequence of Proposition 2. The upper bound for $d_1 \ge 1$ follows from Proposition 1. Note that Proposition 1 gives an upper bound for the $L_2(\Omega, \mu)$ error, which is, of course, also an upper bound for the $L_1(\Omega, \mu)$ error used in the definition of e_n^{ran}. It remains to verify the upper bound in case $d_1 = 0$. This, however, is just (weighted) integration of f:

$$T_k f = \int_Q k(0, y) f(y) dy,$$

and its randomized approximation is well-known. Indeed, consider it as integration of the function $k(0, y) f(y) \chi_Q(y)$ over $B(0, \bar{\varrho})$, where $\bar{\varrho} = \sup\{|y| : y \in Q\}$. Using the standard Monte Carlo method with importance sampling with n samples of density $p(y) = a^{-1} \gamma_\sigma(0, y)$, where

$$a = \int_{B(0,\bar{\varrho})} \gamma_\sigma(0, y) dy$$

(this is just the $\varphi^{(0)}$ approximation from section 3, that is, the algorithm with $m = 0$), it follows readily that the expected mean square error is $\le c n^{-1/2} \|k\|_{\mathcal{C}^{s,\sigma}}$. □

For the next result we specify $F = B_{\mathcal{C}^{s,\sigma}(M,Q)} \times \mathcal{B}_{\mathcal{L}_\infty(Q)}$, $G = L_\infty(M)$ (replaced by $\mathbb{R}$, if $d_1 = 0$), the solution operator S is given by $S(k,f) = T_k f$, and

$$\Lambda = \left\{ \delta^\alpha_{(x,y)} : (x,y) \in M \times Q \setminus \operatorname{diag}(M,Q),\ \alpha \in \mathbb{N}_0^{d_1},\ |\alpha| \le s \right\} \cup \{\delta_y : y \in Q\},$$

where $\delta^\alpha_{(x,y)}(k,f) = D^\alpha_x k(x,y)$ and $\delta_y(k,f) = f(y)$. We define α_2 as

$$\alpha_2 = \begin{cases} \frac{s}{d_1} & \text{if} \quad \frac{s}{d_1} \le \min\left(\frac{d+\sigma}{d_1}, \frac{1}{2}\right) \\ \frac{1}{2} & \text{if} \quad \frac{s}{d_1} > \min\left(\frac{d+\sigma}{d_1}, \frac{1}{2}\right) \qquad \text{and} \quad d_1 = d = -2\sigma \\ 0 & \text{otherwise.} \end{cases} \tag{66}$$

Proposition 3. *Let $0 \le d_1 \le d$ and assume that Q has non-empty interior. Then there is a constant $c > 0$ such that for all $n \in \mathbb{N}$ with $n \ge 2$ the following holds:*

$$e_n^{\text{ran}}(S,F) \ge c n^{-\min\left(\frac{s}{d_1}, \frac{d+\sigma}{d_1}, \frac{1}{2}\right)} (\log n)^{\alpha_2}.$$

Proof. First we consider the case

$$\frac{s}{d_1} \le \min\left(\frac{d+\sigma}{d_1}, \frac{1}{2}\right).$$

Let x_0 be any inner point of Q, let Q', be any cube of the form $Q' = x_0 + \delta_1[-\frac{1}{2}, \frac{1}{2}]^d$ contained in Q. Let f_0 be the function on Q which is identically equal to 1. Define $R_1 : C_0^s(M \times Q') \to \mathcal{C}^{s,\sigma}(M,Q) \times \mathcal{L}_\infty(Q)$ by setting $R_1(g) = (k, f_0)$ for $g \in C_0^s(M \times Q')$, where

$$k(x,y) = \begin{cases} c_1 g(x,y) & \text{if} \quad y \in Q' \\ 0 & \text{otherwise,} \end{cases}$$

for $(x,y) \in M \times Q$, and

$$c_1 = \inf\{\gamma_{\sigma-l}(x,y) : 0 \le l \le s,\ x \in M,\ y \in Q', x \ne y\}.$$

Put $F_1 = B_{C_0^s(M \times Q')}$, $G_1 = L_\infty(M)$, and

$$\Lambda_1 = \{\delta^\alpha_{(x,y)} : (x,y) \in M \times Q', \alpha \in \mathbb{N}_0^{d_1+d}, |\alpha| \le s\},$$

where $\delta^{\alpha}_{(x,y)} g(x,y) = D^{\alpha} g(x,y)$ is the partial derivative with respect to the variables x and y. Define $S_1 : C_0^s(M \times Q') \to L_\infty(M)$ by

$$(S_1 g)(x) = \int_{Q'} g(x,y)dy \quad (x \in M)$$

for $g \in C_0^s(M \times Q')$. We have

$$c_1^{-1}(S \circ R_1(g))(x) = c_1^{-1} \int_Q k(x,y) f_0(y)dy = \int_{Q'} g(x,y)dy = (S_1 g)(x),$$

R_1 maps $B_{C_0^s(M \times Q')} = F_1$ to $B_{\mathcal{C}^{s,\sigma}(M,Q)} \times \mathcal{B}_{\mathcal{L}_\infty(Q)} = F$, and is of the form (51). Therefore, by Lemma 7,

$$e_n^{\mathrm{ran}}(S_1, B_{C_0^s(M \times Q')}) \le c_1^{-1} e_n^{\mathrm{ran}}(S, F).$$

Since $s/d_1 \le 1/2$, [10], Prop. 5.1 gives

$$e_n^{\mathrm{ran}}(S_1, B_{C_0^s(M \times Q')}) \ge c n^{-s/d_1} (\log n)^{s/d_1} \tag{67}$$

(the related lower bound proof also holds for functions which satisfy the boundary conditions, and for $L_\infty(M)$ instead of $C(M)$ as a target space). Consequently,

$$e_n^{\mathrm{ran}}(S, F) \ge c n^{-s/d_1} (\log n)^{s/d_1}.$$

Now we assume

$$\frac{s}{d_1} > \min\left(\frac{d+\sigma}{d_1}, \frac{1}{2}\right)$$

and use Proposition 2 for a reduction. Put

$$k(x,y) = \begin{cases} |x-y|^\sigma & \text{if} \quad \sigma \le d_1/2 - d \\ 1 & \text{if} \quad \sigma > d_1/2 - d \end{cases} \qquad (x \in M, y \in Q, x \ne y).$$

Let $k_0 = \|k\|^{-1}_{\mathcal{C}^{s,\sigma}} k$, define

$$R_2 : \mathcal{L}_\infty(Q) \to \mathcal{C}^{s,\sigma}(M,Q) \times \mathcal{L}_\infty(Q)$$

by

$$R_2(f) = (k_0, f)$$

and let $S_2 = T_{k_0}$. We set $F_2 = \mathcal{B}_{\mathcal{L}_\infty(Q)}$, $G_2 = L_\infty(M)$, and $\Lambda_2 = \{\delta_y : y \in Q\}$. Then $S_2 = S \circ R_2$, $R_2(F_2) \subseteq F$, and R_2 is of the form (51). It follows from Lemma 7 and Proposition 2 that

$$e_n^{\mathrm{ran}}(S, F) \ge c n^{-\min\left(\frac{d+\sigma}{d_1}, \frac{1}{2}\right)} (\log n)^{\alpha_1}.$$

$\square$

As a consequence of Propositions 1 and 3 we get matching, up to logarithmic factors, upper and lower bounds for the minimal error $e_n^{\rm ran}(S,F)$, with α_2 and β defined in (66) and (25), respectively.

Theorem 2. *Let $0 \le d_1 \le d$, let $\sigma \in \mathbb{R}$, $-d < \sigma < +\infty$, let M, Q be as defined in section 2, assume that Q has non-empty interior, and let $s \in \mathbb{N}$. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ with $n \ge 2$ the following holds:*

$$cn^{-\min\left(\frac{s}{d_1},\frac{d+\sigma}{d_1},\frac{1}{2}\right)}(\log n)^{\alpha_2} \le e_n^{\rm ran}(S,F) \le n^{-\min\left(\frac{s}{d_1},\frac{d+\sigma}{d_1},\frac{1}{2}\right)}(\log n)^{\beta}.$$

5 An application to Poisson's equation

We study a simple prototype problem, the randomized complexity of which nevertheless has been left open so far: Let $d \ge 2$ and $0 \le d_1 \le d$. Let $Q \subset \mathbb{R}^d$ be the d-dimensional (Euclidean) unit ball around zero and let M be a d_1-dimensional cube, contained in the interior of Q, that is, if $d_1 \ge 1$,

$$M = x_0 + a[0,1]^{d_1} \times \{0^{(d-d_1)}\} \subset Q^0,$$

where $a > 0$, and $M = \{x_0\} \subset Q^0$ if $d_1 = 0$. For $f \in \mathcal{L}_\infty(Q)$ let $u \in C(Q)$ be the (generalized) solution of

$$-\Delta u = f \qquad u|_{\partial Q} = 0. \tag{68}$$

Define $S_1 : \mathcal{L}_\infty(Q) \to L_\infty(M)$ as

$$S_1 f = u|_M,$$

that is, given $f \in \mathcal{L}_\infty(Q)$, we want to compute the solution u on a d_1-dimensional subcube, the error measured in the norm of $L_\infty(M)$. So here we put $F = \mathcal{B}_{\mathcal{L}_\infty(Q)}$, $G = L_\infty(M)$, and $\Lambda = \{\delta_y : y \in Q\}$ (in the case $d_1 = 0$ we replace $L_\infty(M)$ by $\mathbb{R}$). The Green's function for problem (68) is explicitly known:

$$k(x,y) = \begin{cases} \frac{1}{(d-2)c(d)}\left(\frac{1}{|x-y|^{d-2}} - \frac{1}{(|y||x-\bar{y}|)^{d-2}}\right) & \text{if} \quad d \ge 3 \\ -\frac{1}{2\pi}\left(\ln|x-y| - \ln(|y||x-\bar{y}|)\right) & \text{if} \quad d = 2. \end{cases} \tag{69}$$

and is defined for all $x, y \in Q$ with $x \ne y$. Here $\bar{y} = y/|y|^2$ and $c(d)$ is the surface measure of the unit sphere in $\mathbb{R}^d$. If $y = 0$, then the second term in the brackets on the right-hand side is replaced by 1 for $d \ge 3$ and by

0 for $d = 2$. So the solution to (68) is given by $T_k f$, or, in other words, $S_1 = T_k$. We have $k \in \mathcal{C}^{s,2-d}(M,Q)$ for all $s \geq 1$. Indeed, since the closed set M is contained in the open set Q^0 and $\bar{y}$ is in the complement of Q^0, the respective second term on the right-hand side of (69) is in $C^{s,\sigma}(M,Q)$ for all $s \in \mathbb{N}$ and $\sigma > 0$. That the first term belongs to $\mathcal{C}^{s,2-d}(M,Q)$ for all $s \in \mathbb{N}$ was already discussed in Section 2. With $\sigma = 2 - d$ and any $s > d + \sigma = 2$, the exponents defined in (53) and (25) become

$$\alpha_1 = \begin{cases} \frac{1}{2} & \text{if} \quad d_1 = d = 4 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\beta = \begin{cases} 0 & \text{if} \quad d_1 < 4 \\ \frac{3}{2} & \text{if} \quad d_1 = 4 \\ \frac{2}{d_1} & \text{if} \quad d_1 > 4. \end{cases}$$

Theorem 3. *Let M, Q be as above. Then there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ with $n \geq 2$,*

$$c_1 n^{-\min\left(\frac{2}{d_1},\frac{1}{2}\right)} (\log n)^{\alpha_1} \leq e_n^{\mathrm{ran}}(S_1, \mathcal{B}_{\mathcal{L}_\infty(Q)}) \leq c_2 n^{-\min\left(\frac{2}{d_1},\frac{1}{2}\right)} (\log n)^{\beta}.$$

Proof. This is a direct consequence of Theorem 1. □

6 Comments

Parts of this paper have already been presented in a talk at the Dagstuhl Seminar "Algorithms and Complexity of Continuous Problems" 2000, see [7].

Kollig and Keller [12] used a one-level splitting like (15) to develop an algorithm for solving the rendering integral equation providing the global illumination of scenes in computer graphics. They report good numerical test results.

We considered only the simplest case of M being a cube. Clearly, the analysis carries over to finite unions of cubes, to simplices and their finite unions, and other domains on which suitable approximation tools are available. In [9] we show that the case of the cube is sufficient to handle general C^∞ domains by introducing local charts.

Although we dealt only with real-valued k and f, the results generalize in an obvious way to the complex case.

The function classes related to f did not possess any smoothness. Classes of finite smoothness are considered in [9].

The results of section 5 are generalized in [9]. There the information complexity of general elliptic PDE with smooth coefficients and in smooth domains is treated.

Let us compare the rates obtained here with those in the deterministic setting. By simple reduction to integration one can show that under the assumptions of Theorem 1 there are constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}_0$,

$$c_1 \le e_n^{\det}(T_k, \mathcal{B}_{\mathcal{L}_\infty(Q)}) \le c_2,$$

similarly, under the assumptions of Theorem 2 ($F = B_{\mathcal{C}^{s,\sigma}(M,Q)} \times \mathcal{B}_{\mathcal{L}_\infty(Q)}$),

$$c_1 \le e_n^{\det}(S, F) \le c_2,$$

and of Theorem 3,

$$c_1 \le e_n^{\det}(S_1, \mathcal{B}_{\mathcal{L}_\infty(Q)}) \le c_2,$$

meaning that for the function classes considered here no deterministic algorithm can give a non-trivial convergence rate.

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